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Abstract

We prove the uniform AP convergence on unstructured meshes in 2D of a generalization, see [7], of the Gosse-Toscani 1D scheme for the hyperbolic heat equation. This scheme is also a nodal extension in 2D of the Jin-Levermore scheme described in [23] for the 1D case. In 2D, the proof is performed using a new diffusion scheme.

1 Introduction

We address the convergence analysis on unstructured meshes of diffusion asymptotic preserving schemes for the discretization of a problem with a stiff parameter denoted as $0 < \varepsilon \leq 1$. The model problem considered in this work is the hyperbolic heat equation in the domain $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^n$

$$P^\varepsilon : \qquad \begin{cases} \partial_t p^\varepsilon + \dfrac{1}{\varepsilon}\operatorname{div}(\mathbf{u}^\varepsilon) = 0, & p^\varepsilon \in \mathbb{R}, \\ \partial_t \mathbf{u}^\varepsilon + \dfrac{1}{\varepsilon}\nabla p^\varepsilon = -\dfrac{\sigma}{\varepsilon^2}\mathbf{u}^\varepsilon, & \mathbf{u}^\varepsilon \in \mathbb{R}^n \end{cases} \tag{1}$$

discretized with first order finite volume schemes. This problem is representative of many transport problem such as transfer and neutron transport, for which the small parameter ε is the ratio of two very different sound velocities and σ is the absorption or the opacity. For simplicity both ε and $\sigma > 0$ are kept constant in space in this study. The system (1) can also be introduced as a specific linearization of a pressure-velocity system of partial differential equations in the acoustic regime. In this work we will need the following well known energy estimates concerning the solution $\mathbf{V}^\varepsilon$ of the Cauchy problem for the partial differential equation (1).

Proposition 1.1. *If $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$, then*

$$||\mathbf{V}^\varepsilon||_{H^p(\Omega)} \leq ||\mathbf{V}^\varepsilon(0)||_{H^p(\Omega)} \tag{2}$$

and moreover

$$\frac{\sigma}{\varepsilon^2}||\mathbf{u}^\varepsilon||^2_{L^2([0,T];H^p(\Omega))} \leq ||\mathbf{V}^\varepsilon(0)||^2_{H^p(\Omega)}. \tag{3}$$

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We will consider well prepared data in the sense that: $p^\varepsilon(t=0)$ is independent of ε and is sufficiently smooth; the initial velocity satisfies the equality in the second equation of (1) at leading order. It writes

$$p^\varepsilon(t=0) = p_0 \text{ and } \mathbf{u}^\varepsilon(0) = -\frac{\varepsilon}{\sigma}\nabla p_0. \tag{4}$$

For such well prepared data, it can be easily shown that the formal limit of P^ε for small ε is

$$P^0: \qquad \partial_t p - \frac{1}{\sigma}\Delta p = 0. \tag{5}$$

Remark 1.2. *We do not consider the regime $\sigma \to 0$, since it introduces a singularity both in the initial data of the hyperbolic heat equation and in the limit parabolic equation.*

1.1 Precision of AP discretizations

Before addressing the main difficulty of this work which is the discretization on unstructured meshes, we briefly recall the now well known notion of an asymptotic preserving technique [21]-[22] which is illustrated in the figure 1. For the simplicity of the presentation, we will consider mainly semi-discrete numerical methods, this is why the time step does not show up in the graphic. The parameter h designs a numerical method with characteristic length $h \leq 1$: that is we assume a numerical method P_h^ε for the discretization of P^ε.

Definition 1.1 (Uniform AP)**.** *If P_h^ε is consistent with P^ε uniformly with respect to ε, then we say that the scheme P_h^ε is uniformly AP (uniformly asymptotic preserving).*

However the design of such methods and the numerical proof of this property is difficult. This is why it has been proposed in [21] to rely on the simpler necessary condition, where the limit as $\varepsilon \to 0$ of P_h^ε is called the limit diffusion scheme P_h^0.

$$\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\
P^\varepsilon & P^0 \arrow[l, "\varepsilon \to 0"]
\end{tikzcd}$$

Figure 1: The AP (asymptotic diagram) diagram

Definition 1.2 (AP)**.** *If P_h^0 is consistent with the limit model P^0, then we say that the scheme P_h^ε is AP (asymptotic preserving).*

This property is simpler to analyze than the uniform AP. It explains why it has been very fruitful in the past. In 1D, many AP schemes have been designed for some PDE and physical problems: S. Jin, C. D. Levermore [23] or L. Gosse, G. Toscani [18] for the hyperbolic heat equation, M. Lemou, L. Mieussens, N. Crouseilles [27]-[10]-[11] for some kinetic equations, L. Gosse [19], C. Buet and co-workers [6] or S. Jin and C. D. Levermore [24] for S_N equations and C. Berthon, R. Turpault [2]-[3]-[4] for generic systems and a non linear radiative transfer model. Recently some asymptotic preserving schemes for linear systems and non linear radiative transfer model have been designed in 2D [7]-[8]-[9]. Other application to non linear hyperbolic systems of conservation laws with stiff diffusive relaxation is to be found is [30]. Relaxation systems are treated in [15]. More general situation for transport and discrete velocity systems are in [25, 26]. However for this type of schemes it is difficult to obtain convergence estimates due to the competition between the two parameters ε and h. To our knowledge this type of proof are only given for uniform grids [7] (consistence and stability, Lax theorem), [18] (L^1 and BV estimates), [28] (L^2 estimates). The goal of this work is to prove the uniform AP property on unstructured grids.

To this end we adapt a strategy developed in [16] in a slightly different context. It relies on the derivation of a priori estimates attached to the AP diagram in figure 1. To have a more global perspective on this strategy, let us assume some natural abstract a priori estimates for a given norm which is in our work based on $\|f\| = \|f\|_{L^2([0,T]\times\Omega)}$ or $\|f\| = \|f\|_{L^\infty([0,T];L^2(\Omega))}$ where $T > 0$ is a given final time, $\Omega = \mathbb{R}$, in 1D or $\Omega = [0,1]^2$ with periodic boundary conditions in 2D. We assume five constants $a, b, c, d, e > 0$ and four additional constants ${}_{\downarrow}C, C^{\rightarrow}, C_{\leftarrow}, C_{\downarrow} > 0 > 0$ such that the error attached to the branches of the AP diagram can be bounded like

$$\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}} \le {}_{\downarrow}C\varepsilon^{-b}h^c, \tag{6}$$

$$\|P_h^\varepsilon - P_h^0\| \le C^{\rightarrow}\varepsilon^e. \tag{7}$$

$$\|P_h^0 - P^0\| \le C_{\downarrow}h^d, \tag{8}$$

$$\|P^\varepsilon - P^0\| \le C_{\leftarrow}\varepsilon^a, \tag{9}$$

The first inequality is the naive error bound which typically blows up for small ε. The second inequality for $\|P_h^\varepsilon - P_h^0\|$ is assumed to have a form similar to the last one which expresses that P^0 is the limit of P^ε. The third inequality corresponds to the usual AP property.

Proposition 1.3. *Assume that all these inequalities are at hand and that $d \ge c$ and $e \ge a$. Then the uniform AP holds with a rate at least $O\left(h^{\frac{ac}{a+b}}\right)$.*

Proof. The triangular inequality writes

$$\|P_h^\varepsilon - P^\varepsilon\| \le \min\left(\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}, \|P_h^\varepsilon - P_h^0\| + \|P_h^0 - P^0\| + \|P^\varepsilon - P^0\|\right)$$

which yields, using $\min(x, y+z) \le \min(x,y) + \min(x,z)$, $d \ge c$ and $e \ge a$,

$$\|P_h^\varepsilon - P^\varepsilon\| \le C\left(\min\left(\varepsilon^{-b}h^c, \varepsilon^e\right) + h^d + \min\left(\varepsilon^{-b}h^c, \varepsilon^a\right)\right) \le C\left(2\min\left(\varepsilon^{-b}h^c, \varepsilon^a\right) + h^d\right) \tag{10}$$

with $C = \max\left({}_{\downarrow}C, C^{\rightarrow}, C_{\downarrow}, C_{\leftarrow}\right)$. We define a threshold value $\varepsilon_{\text{thresh}}$ by $\varepsilon_{\text{thresh}}^{-b}h^c = \varepsilon_{\text{thresh}}^a$. So either $\varepsilon \le \varepsilon_{\text{thresh}}$ so that

$$\min\left(\varepsilon^{-b}h^c, \varepsilon^a\right) \le \varepsilon_{\text{thresh}}^a = h^{\frac{ac}{a+b}},$$

or $\varepsilon \ge \varepsilon_{\text{thresh}}$ and the same bound is obtained by taking the other term as the minimum. And since $d \ge c$, one gets the abstract bound $\|P_h^\varepsilon - P^\varepsilon\| \le 3Ch^{\frac{ac}{a+b}}$ which ends the proof. ☐

1.2 Organization of the proof

The structure of these inequalities explains our strategy: that is we prove separately each of these inequalities (9-7) with care, so that the inequalities $d \ge c$ and $e \ge a$ are true. This part of the proof relies on specific hyperbolic and parabolic numerical methods. Even if it is technical, the first three inequalities (9-8) do not yield additional difficulties with respect to the state of the art. The proof of inequality (7) is provided in 1D, and can be probably be generalized straightforwardly on cartesian meshes in 2D and 3D. On the other hand our researches on proving (7) for $\|P_h^\varepsilon - P_h^0\|$ show a fundamental obstruction in dimension greater than one on unstructured meshes which was not expected initially. Since the main difficulty is related to P_h^0, it motivates the definition of a new diffusion scheme. To this end we remark that another diffusion scheme is naturally defined from P_h^ε by killing the derivative $\partial_t v_h$ in the discrete version of the second equation of (1). Killing at the continuous level the $\partial_t v$ is absolutely equivalent to taking the formal limit $\varepsilon \to 0^+$.

But at the discrete level, it appears that it generates a new family of diffusion schemes, where both parameters h and ε are present. We call them Diffusion Asymptotic schemes, DA_h^ε. By construction $P_h^0 = \lim_{\varepsilon\to 0} DA_h^\varepsilon$. This is summarized in figure 2. Finally since the scheme DA_h^ε is still an accurate discretization of P^0, our proof of the uniform AP property is based on the new AP diagram displayed in figure 3.

$$P_h^\varepsilon \xrightarrow{\quad \partial_t v_h = 0 \quad} DA_h^\varepsilon \xrightarrow{\quad \varepsilon \to 0 \quad} P_h^0$$

Figure 2: Definition of the diffusion asymptotic scheme DA_h^ε.

Our main theorem 3.16 in dimension 2 is based on this structure and it may be stated as follows: **The so-called JL-(b) scheme defined in [7] for the discretization of the hyperbolic heat equation (1) (the scheme is cell-centered with nodal based fluxes) is uniformly AP on unstructured meshes, with a rate of convergence at least $O(h^{\frac{1}{4}})$ for sufficiently smooth initial data**. This is an improvement with respect to [7] where only AP was proven. To our knowledge this is the first time that such a result is obtained on general unstructured multidimensional meshes. More precisely the convergence estimate can be written as

$$\textbf{error} \le C \min\left(\sqrt{\frac{h}{\varepsilon}} \;\; , \;\; \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + h + \varepsilon \right)$$

where the first argument in the min function comes from the hyperbolic analysis and the second argument comes from the parabolic analysis. Some natural regularity assumptions are nevertheless imposed on the mesh in the hypothesis 2.1, this is not very restrictive. For example meshes with angles greater than 90 degrees are allowed. If the mesh is made with triangles, the hypothesis is fulfilled if all angles are greater than 12 degrees, see [7]. It is interesting to notice that the rate of uniform convergence is $O(h^{\frac{1}{3}})$ in dimension one. The difference essentially comes from the estimate of the reconstruction of the initial velocity which is needed to rewrite a diffusion scheme as a non homogeneous hyperbolic scheme: it is much simpler in dimension one (see equation (23)) than in dimension two (see proposition (3.13)). In this work we considered mainly semi-discrete numerical schemes, since it simplifies a lot the notations and allow to focus on the main difficulties, but the final estimates of convergence can be generalized to fully discrete schemes, using the a priori estimates developed in [12]. For explicit schemes, these estimates add a term proportional to the square root of the maximal time step allowed by the CFL condition. Since our problem is an hyperbolic+relaxation problem, with a limit which is parabolic, this additional term can be computed and is of the order between h (for purely hyperbolic) to h^2 (for purely parabolic). We refer to [7] for the detail of CFL condition in 1D and 2D. Concerning the implicit fully discrete version of the semi-discrete scheme which is unconditionally stable and well adapted to the test problem analyzed at the end of this work, the same kind of error terms can be analyzed. We will obtained the following result in dimension two.

Theorem 1.1. *With some usual regularity assumptions on the mesh, the error between our cell-centered finite volume corner-based-flux implicit discretization $\mathbf{P}_{h,\Delta t}^\varepsilon$ and the exact solution is*

$$\|\mathbf{V}_h^\varepsilon(t_n) - \mathbf{V}^\varepsilon(t_n)\|_{L^2(\Omega)} \le C\left(h^{\frac{1}{4}} + \Delta t^{\frac{1}{2}}\right)\|p_0\|_{H^4(\Omega)}, \qquad t_n = n\Delta t \le T.$$

The constant is independent of h, ε and Δt and behaves less than $T^{\frac{3}{2}}$ for large T.

The proof is an easy add-on on the space estimate $O(h^{\frac{1}{4}})$ of theorem 3.16, by means of an abstract method [12] which gives a general bound $O(\Delta t^{\frac{1}{2}})$ of the difference between the semi-discrete scheme and the implicit Euler scheme. This will be explained at the end of this work. The rate of convergence is confirmed by the numerical results of section 5, which show an even better rate of convergence.

$$
\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\partial_t \mathbf{v}_h = 0"] \arrow[d, "h \to 0"'] & DA_h^\varepsilon \arrow[r, dashed, "\varepsilon \to 0"] \arrow[d, "h \to 0 \text{ and } \varepsilon \to 0"] & P_h^0 \arrow[dl, dashed] \\
P^\varepsilon & P^0 \arrow[l, "\varepsilon \to 0"'] &
\end{tikzcd}
$$

Figure 3: The new AP diagram, where the previous branch is still displayed in dashed lines.

We think that some of our results can have an interest for the development and use of such methods in research or industrial codes with complex non linear physics on unstructured meshes. Indeed for such codes cell-centered Finite Volume schemes are a natural solution in terms of data structure. The point is the following: the scheme studied in this work is the only cell-centered one we know in 2D to compute the solutions of problems which admit diffusion limits in certain regimes and for which it is possible to prove the AP property. Since the structure of this cell-centered scheme is nodal based, it strongly questions the ability of standard Finite Volume methods with edge-based fluxes to recover asymptotic diffusion regimes. As demonstrated in this work, nodal based Finite Volume techniques do not suffer from this drawback. For linear wave equation the nodal scheme can be understand as some 1D Riemann problem written in some direction around each node, so can be interpreted as an approximation of the 2D Riemann problem [17].

1.3 Organization of the work

This work is organized as follows. Section 2 is dedicated to the discretization of the model problem in dimension one on irregular grids. The convergence is proved in theorem 2.10 with order $h^{\frac{1}{3}}$ in the L^2 space-time norm. In the next section, the nodal solvers for the hyperbolic equation are defined, and the various a priori estimates proved. The main theorem of uniform AP for the JL-(b) scheme with a rate $O(h^{\frac{1}{4}})$ is proved at the end of the section. Section 5 provides numerical results that sustain the fact that the convergence order depends on the relative value of ε and h, and so is mixed hyperbolic/parabolic. Our final remarks will be gathered in a conclusion. All our results and numerical methods in 2D can be generalized in 3D provided a convenient definition of the nodal corner vector is used as in [13].

2 Analysis in 1D

The model problem in dimension one writes

$$
P^\varepsilon : \qquad \begin{cases} \partial_t p^\varepsilon + \frac{1}{\varepsilon}\partial_x u^\varepsilon = 0, \\ \partial_t u^\varepsilon + \frac{1}{\varepsilon}\partial_x p^\varepsilon = -\frac{\sigma}{\varepsilon^2} u^\varepsilon. \end{cases} \tag{11}
$$

As stressed already in (4), we consider well-prepared data $p^\varepsilon(t=0) = p_0$ and $u_0^\varepsilon = -\frac{\varepsilon}{\sigma}\partial_x p_0$. The equations (11) admit the formal diffusion limit when ε tends to 0:

$$
P^0 : \qquad \partial_t p - \frac{1}{\sigma}\partial_{xx} p = 0. \tag{12}
$$

A useful variable will be the scaled gradient

$$
v = -\frac{1}{\sigma}\partial_x p. \tag{13}
$$

2.1 Notations

We denote $x_{j+1/2}$ the nodes, the cells j are the intervals $[x_{j-1/2}, x_{j+1/2}]$, thus $\Delta x_j = x_{j+1/2} - x_{j-1/2}$, x_j is the center of the cell j that is $x_j = \frac{1}{2}(x_{j+1/2} + x_{j-1/2})$, and $\Delta x_{j+1/2} = x_{j+1} - x_j = \frac{1}{2}(\Delta x_{j+1} + \Delta x_j)$. Natural assumptions on the mesh are summarized below:

Hypothesis 2.1 (Regularity of the mesh in 1D and constant $C_{\mathcal{M}}$)**.** *We consider that there exists a universal constant $0 < C_{\mathcal{M}} \leq 1$ independent of the mesh size $h = \sup_{j\in\mathbb{Z}} \Delta x_j$ which controls the mesh from below*

$$C_{\mathcal{M}} h \leq \Delta x_j \leq h \qquad \forall j \in \mathbb{Z}. \tag{14}$$

The semi-discrete JL(b) scheme, derived in [7] in 2D, can also be written in 1D on irregular meshes as

$$P_h^\varepsilon : \quad \begin{cases} \dfrac{d}{dt} p_j^\varepsilon + \dfrac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = 0, \\ \dfrac{d}{dt} u_j^\varepsilon + \dfrac{p_{j+\frac{1}{2}}^\varepsilon - p_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = -\dfrac{\sigma}{\varepsilon^2} \dfrac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2}, \end{cases} \tag{15}$$

with the fluxes $p_{j+\frac{1}{2}}^\varepsilon$ and $u_{j+\frac{1}{2}}^\varepsilon$ are the solutions of the well-posed linear system

$$j \in \mathbb{Z} : \quad \begin{cases} p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \dfrac{\sigma \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}^\varepsilon & = p_j^\varepsilon + u_j^\varepsilon, \\ -p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \dfrac{\sigma \Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}^\varepsilon & = -p_{j+1}^\varepsilon + u_{j+1}^\varepsilon. \end{cases} \tag{16}$$

This scheme is the same as the Gosse-Toscani scheme[1]. Other equivalent forms of P_h^ε can be obtained by various manipulations, as in (29). We use another formulation of the Gosse-Toscani obtained using the Jin-Levemore scheme [23] and a discretization of the source term which uses the fluxes. Contrary to the Gosse-Toscani scheme which uses Riemann problem, this formulation based an elementary algebraic computation is easier to write in 2D on unstructured meshes (the design is detailed in [7]). The natural pointwise initialization is chosen

$$p_j^\varepsilon(0) = p_0(x_j) \text{ and } u_j^\varepsilon(0) = -\frac{\varepsilon}{\sigma} \partial_x p_0(x_j) \text{ for all } j \in \mathbb{Z}. \tag{17}$$

[1] A long and tedious computation shows that the scheme is strictly equivalent to the Gosse-Toscani's scheme, described in [18] but only for uniform meshes, which writes in terms of $w^\varepsilon, v^\varepsilon = p^\varepsilon \pm u^\varepsilon$

$$\begin{cases} \dfrac{dw_j}{dt} + \dfrac{M_{j-\frac{1}{2}}}{\varepsilon} \dfrac{w_j^\varepsilon - w_{j-1}^\varepsilon}{\Delta x_j} = \dfrac{1}{\varepsilon \Delta x_j}(1 - M_{j-\frac{1}{2}})(v_j^\varepsilon - w_j^\varepsilon) = M_{j-\frac{1}{2}} \dfrac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} \dfrac{\sigma}{2\varepsilon^2}(v_j^\varepsilon - w_j^\varepsilon), \\ \dfrac{dv_j^\varepsilon}{dt} - \dfrac{M_{j+\frac{1}{2}}}{\varepsilon} \dfrac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\Delta x_j} = \dfrac{1}{\varepsilon \Delta x_j}(1 - M_{j+\frac{1}{2}})(w_j^\varepsilon - v_j^\varepsilon) = M_{j+\frac{1}{2}} \dfrac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} \dfrac{\sigma}{2\varepsilon^2}(w_j^\varepsilon - v_j^\varepsilon) \end{cases}$$

with $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{\sigma \Delta x_{j+\frac{1}{2}} + 2\varepsilon}$and $\Delta x_{j+\frac{1}{2}} = \frac{\Delta x_j + \Delta x_{j+1}}{2}$. By writing

$$\begin{cases} M_{j-\frac{1}{2}}(w_{j-1}^\varepsilon - w_j^\varepsilon) = M_{j-\frac{1}{2}} w_{j-1} - M_{j+\frac{1}{2}} w_j + (M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}) w_j^\varepsilon \\ M_{j+\frac{1}{2}}(v_{j+1}^\varepsilon - v_j^\varepsilon) = M_{j+\frac{1}{2}} v_{j+1}^\varepsilon - M_{j-\frac{1}{2}} v_j^\varepsilon - (M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}) v_j^\varepsilon \end{cases}$$

then in terms of p^ε an u^ε we have evidently

$$\begin{cases} \dfrac{dp_j^\varepsilon}{dt} + \dfrac{1}{\varepsilon} \dfrac{M_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^\varepsilon - M_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^\varepsilon}{\Delta x_j} = 0, \\ \dfrac{du_j^\varepsilon}{dt} + \dfrac{1}{\varepsilon} \dfrac{M_{j+\frac{1}{2}} p_{j+\frac{1}{2}}^\varepsilon - M_{j-\frac{1}{2}} p_{j-\frac{1}{2}}^\varepsilon}{\Delta x_j} = -\dfrac{1}{2} \left(M_{j+\frac{1}{2}} \dfrac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} \dfrac{\sigma}{\varepsilon^2} + M_{j-\frac{1}{2}} \dfrac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} \dfrac{\sigma}{\varepsilon^2} \right) u_j^\varepsilon + \dfrac{M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} p_j^\varepsilon \end{cases}$$

with the fluxes given by $p_{j+\frac{1}{2}}^\varepsilon = \frac{p_j^\varepsilon + p_{j+1}^\varepsilon}{2} + \frac{u_j^\varepsilon - u_{j+1}^\varepsilon}{2}$ and $u_{j+\frac{1}{2}} = \dfrac{u_j^\varepsilon + u_{j+1}^\varepsilon}{2} + \dfrac{p_j^\varepsilon - p_{j+1}^\varepsilon}{2}$.

When ε tends to 0, the scheme P_h^ε admits the diffusion limit scheme P_h^0

$$P_h^0: \qquad \Delta x_j \frac{d}{dt} p_j - \frac{1}{\sigma}\left(\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \frac{p_j - p_{j-1}}{\Delta x_{j-\frac12}} \right) = 0 \tag{18}$$

with the pointwise initialization

$$p_j(0) = p_0(x_j) \text{ for all } j \in \mathbb{Z}. \tag{19}$$

Other quantities are the reconstructed gradient

$$\begin{cases} v_{j+\frac12} = -\dfrac{1}{\sigma} \dfrac{p_{j+1}-p_j}{\Delta x_{j+\frac12}}, \\ v_j = \dfrac{v_{j+\frac12} + v_{j-\frac12}}{2}. \end{cases} \tag{20}$$

We denote by $\mathbf{V}^\varepsilon(t,x) = (p^\varepsilon(x,t), u^\varepsilon(x,t))$ the solution of the hyperbolic heat equations P^ε. We reconstruct similar quantities from the diffusion scheme: it yields $\mathbf{W}^\varepsilon(t,x) = (p(x,t), \varepsilon v(x,t))$ which is the solution of the diffusion limit (12)-(13). The indicatrix function of the interval $(x_{j-1/2}, x_{j+1/2})$ is denoted as $1_j(x) = 1$ if $x \in (x_{j-1/2}, x_{j+1/2})$ and $1_j(x) = 0$ in the other case. With this notation we note $\mathbf{V}_h^\varepsilon(t,x) = \left(\sum_{j\in\mathbb{Z}} p_j^\varepsilon(t) 1_j(x), \sum_{j\in\mathbb{Z}} u_j^\varepsilon(t) 1_j(x) \right)$ the solution of the JL-(b) scheme P_h^ε. Finally we note $\mathbf{W}_h^\varepsilon(t,x) = \left(\sum_{j\in\mathbb{Z}} p_j(t) 1_j(x), \varepsilon \sum_{j\in\mathbb{Z}} v_j(t) 1_j(x) \right)$ the solution of the diffusion scheme P_h^0 (18)-(20).

For simplicity we choose a final time $T > 0$. All error estimates will be given for $t \le T$, either in the norm $\|f(t)\|_{L^\infty([0,T];L^2(\mathbb{R}))}$, or mostly in the norm $\|f\|_{L^2([0,T]\times\mathbb{R})}$.

Hypothesis 2.2 (Regularity of the initial data and constant $C_\mathcal{A}$)**.** *We consider that there exists a universal constant $C_\mathcal{A} > 0$ which controls all kind of approximations/interpolations/projections on the mesh of exact functions. We will write for example the error estimates at the initial time under the form*

$$\|\mathbf{V}_h^\varepsilon(0) - \mathbf{V}^\varepsilon(0)\|_{L^2(\mathbb{R})} \le C_\mathcal{A} h \|p_0\|_{H^2(\mathbb{R})} \tag{21}$$

and

$$\|\mathbf{W}_h^\varepsilon(0) - \mathbf{W}^\varepsilon(0)\|_{L^2(\mathbb{R})} \le C_\mathcal{A} h \|p_0\|_{H^2(\mathbb{R})}. \tag{22}$$

The second inequality in the hypothesis can be related to the sharper inequality

$$\left\| \sum_j \left(u_j^\varepsilon(0) - \varepsilon v_j(0) \right) 1_j \right\|_{L^2(\mathbb{R})} \le C_\mathcal{A} h \varepsilon \|p_0\|_{H^2(\mathbb{R})}. \tag{23}$$

The other technical constants used to bound the errors of the left, top, right and bottom branches of the AP diagram 1 will be denoted as ${}_\downarrow C$, $C^\rightarrow$, $C_\downarrow$ and $C_\leftarrow$.

2.2 Study of $\|P^\varepsilon - P^0\|$

In this section we prove a natural error estimate [16] between the solution of the hyperbolic heat equations (11) and the solution of the diffusion limit equation (12).

Lemma 2.3. *One has the estimate*

$$\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le C_\leftarrow \varepsilon \|p_0\|_{H^3(\mathbb{R})}, \qquad C_\leftarrow = \frac{T^{\frac32}}{\sigma^2}. \tag{24}$$

Proof. We redefine $v = -\frac{\varepsilon}{\sigma}\partial_x p$ with p the diffusion solution of (12) and introduce R^ε such that the solution of the diffusion equation satisfies

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\partial_x v = 0, \\ \partial_t v + \frac{1}{\varepsilon}\partial_x p + \frac{\sigma}{\varepsilon^2} v = R^\varepsilon \end{cases} \tag{25}$$

where $R^\varepsilon = \partial_t v = -\frac{\varepsilon}{\sigma}\partial_{tx} p = -\frac{\varepsilon}{\sigma^2}\partial_{xxx} p$. Note that $\|R^\varepsilon(t)\|_{L^2(\mathbb{R})} \le \|R^\varepsilon(0)\|_{L^2(\mathbb{R})} \le \frac{\varepsilon}{\sigma^2}\|p_0\|_{H^3(\mathbb{R})}$. Denoting $e^\varepsilon = p - p^\varepsilon$, $f^\varepsilon = v - u^\varepsilon$, we make the difference between the systems (11) et (25)

$$\begin{cases} \partial_t e^\varepsilon + \frac{1}{\varepsilon}\partial_x f^\varepsilon = 0, \\ \partial_t f^\varepsilon + \frac{1}{\varepsilon}\partial_x e^\varepsilon + \frac{\sigma}{\varepsilon^2} f^\varepsilon = R^\varepsilon. \end{cases} \tag{26}$$

Since data are well-prepared, one has $e^\varepsilon(0) = f^\varepsilon(0) = 0$. Consider $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|^2_{L^2(\mathbb{R})} = \|e^\varepsilon\|^2_{L^2(\mathbb{R})} + \|f^\varepsilon\|^2_{L^2(\mathbb{R})}$. Adding the first equation of (26) multiplied by e^ε and the second multiplied by f^ε and integrating on $\mathbb{R}$, we find out that: $\frac{1}{2}\frac{d}{dt}\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|^2_{L^2(\mathbb{R})} \le \int_{\mathbb{R}} R^\varepsilon f^\varepsilon dx \le \|R^\varepsilon\|_{L^2(\mathbb{R})}\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2(\mathbb{R})}$. One gets a bound of $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ by integration between 0 and T. Finally $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le \sqrt{T}\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}))}$ which ends the proof. □

2.3 Stability estimates for P_h^ε and P_h^0

The estimates (27-28) and (31) characterize the dissipation rate of both schemes.

Proposition 2.4. *The scheme P_h^ε is stable in L^2 norm. Moreover,*

$$\sqrt{\int_0^T \left(\sum \Delta x_{j+\frac{1}{2}} (u^\varepsilon_{j+\frac{1}{2}})^2\right) dt} \le \frac{\varepsilon}{\sqrt{\sigma}}\|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})} \tag{27}$$

and

$$\sqrt{\int_0^T \left(\sum_{j\in\mathbb{Z}} (u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_j)^2 + \sum_{j\in\mathbb{Z}} (u^\varepsilon_{j-\frac{1}{2}} - u^\varepsilon_j)^2\right) dt} \le \sqrt{\varepsilon}\|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}. \tag{28}$$

Remark 2.5. *The strategy of the proof of many estimates in this work consists in analyzing the balance between the dissipation of the fluxes and the physical dissipation (all source terms like $-\frac{\sigma}{\varepsilon^2}u$) on the one hand, and some truncation errors on the other hand. This is why it is convenient to reformulate P_h^ε so that the pressure fluxes $p^\varepsilon_{j+\frac{1}{2}}$ and $p^\varepsilon_{j-\frac{1}{2}}$ are eliminated in the second equation of (15). This elimination is technically convenient since all dissipation terms are expressed using the same variable, namely u. It will simplify a lot the comparisons between all kinds of dissipation terms and other errors terms.*

Proof. According to the above remark we obtain the formulation (29) which is equivalent to P_h^ε

$$\begin{cases} \Delta x_j \dfrac{d}{dt} p^\varepsilon_j + \dfrac{u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} = 0, \\ \Delta x_j \dfrac{d}{dt} u^\varepsilon_j - \dfrac{u^\varepsilon_{j+\frac{1}{2}} + u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} + \dfrac{2}{\varepsilon} u^\varepsilon_j = 0, \\ \left(2 + \dfrac{\sigma \Delta x_{j+\frac{1}{2}}}{\varepsilon}\right) u^\varepsilon_{j+\frac{1}{2}} = p^\varepsilon_j - p^\varepsilon_{j+1} + u^\varepsilon_j + u^\varepsilon_{j+1}. \end{cases} \tag{29}$$

Consider now the discrete quadratic energy $E(t) = \frac{1}{2}\sum_j \Delta x_j((p^\varepsilon_j)^2 + (u^\varepsilon_j)^2)$. Multiplying the first equation of (29) by p^ε_j and the second equation by u^ε_j and adding on all the cells, one finds

$$E'(t) = -\sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} p^\varepsilon_j + \sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac{1}{2}} + u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} u^\varepsilon_j - \frac{2}{\varepsilon}\sum_j (u^\varepsilon_j)^2.$$

Since $\sum_j (u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_{j-\frac{1}{2}}) p^\varepsilon_j = \sum_j u^\varepsilon_{j+\frac{1}{2}}(p^\varepsilon_j - p^\varepsilon_{j+1})$, one has by using the third equation of (29) and rearranging the terms

$$E'(t) + \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_j)^2}{\varepsilon} + \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j-\frac{1}{2}} - u^\varepsilon_j)^2}{\varepsilon} + \frac{\sigma}{\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j \frac{(u^\varepsilon_{j+\frac{1}{2}})^2 + (u^\varepsilon_{j-\frac{1}{2}})^2}{2} = 0. \tag{30}$$

Integrating (30) between 0 and t, one finds $E(t) \le E(0)$, that is the L^2 stability of P_h^ε. The estimate (27) comes from $\Delta x_{j+\frac12} = \frac12(\Delta x_j + \Delta x_{j+1})$. The estimate (28) is directly deduced from (30). $\square$

Some similar bounds hold for the quantities related to the diffusion scheme (18). First, multiplying the diffusion scheme by p_j and adding on all the cells, one has the L^2 stability in the sense

$$\frac12 \frac{d}{dt}\sum_j \Delta x_j p_j^2 = -\frac{1}{\sigma}\sum_j \frac{(p_{j+1}-p_j)^2}{\Delta x_{j+\frac12}}.$$

Thus the following estimate holds for the function $\bar v_h = \left(v_{j+\frac12}\right)_j$ defined by (20)

$$\|\bar v_h\|_{L^2([0,T]\times\mathbb{R})} = \sqrt{\int_0^T \sum_j \Delta x_{j+\frac12}(v_{j+\frac12})^2} \le \sqrt{\frac{\sigma}{2}}\|p_h(0)\|_{L^2(\mathbb{R})}, \qquad C>0. \tag{31}$$

2.4 Study of $\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}$

In this section we prove the convergence of P_h^ε to P^ε. We still denote $V^\varepsilon(t) = (p^\varepsilon, u^\varepsilon)$.

Lemma 2.6. *There exists a constant $_\downarrow C > 0$ independent of $h, \varepsilon, C_\mathcal{M}$, with at most a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le \frac{_\downarrow C}{\sqrt{C_\mathcal{M}}}\sqrt{\frac{h}{\varepsilon}}\,\|p_0\|_{H^2(\mathbb{R})}\,. \tag{32}$$

Proof. We use the method introduced by C. Mazeran [29] in his PhD thesis. It starts with an estimate for the time derivative of $\mathscr{E} = \frac12\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|^2_{L^2(\mathbb{R})}$. For the sake of simplicity, q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$ for any quantity q. One has

$$\begin{aligned}\mathscr{E}'(t) = {}& \underbrace{\frac12\int_\mathbb{R}((p_h^\varepsilon)^2 + (u_h^\varepsilon)^2)'dx}_{D_1} + \underbrace{\frac12\int_\mathbb{R}((p^\varepsilon)^2 + (u^\varepsilon)^2)'dx}_{D_2} \\ &+ \underbrace{\int_\mathbb{R}(-(p_h^\varepsilon)'p^\varepsilon - (u_h^\varepsilon)'u^\varepsilon)dx}_{D_3} + \underbrace{\int_\mathbb{R}(-p_h^\varepsilon(p^\varepsilon)' - u_h^\varepsilon(u^\varepsilon)'dx}_{D_4}\end{aligned}$$

We will successively estimate each of those terms, the fundamental idea being that $D_1 \le 0$ and $D_2 \le 0$ are used to control spurious contributions in D_3 and D_4. First D_1 corresponds to the entropy production of the scheme. Thanks to (30), one has

$$D_1 = -\frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}}(u^\varepsilon_{j+\frac12} - u_j^\varepsilon)^2 - \frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}}(u^\varepsilon_{j-\frac12} - u_j^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2}\sum_{j\in\mathbb{Z}}\Delta x_j \frac{(u^\varepsilon_{j+\frac12})^2 + (u^\varepsilon_{j-\frac12})^2}{2} \le 0.$$

One also directly obtains

$$D_2 = -\sum_{j\in\mathbb{Z}}\Delta x_j \frac{\sigma}{\varepsilon^2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}(u^\varepsilon)^2dx\right) \le 0.$$

For D_4, one gets directly

$$D_4 = \sum_{j\in\mathbb{Z}} p_j^\varepsilon \frac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon} + \sum_{j\in\mathbb{Z}} u_j^\varepsilon \frac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} + \sum_{j\in\mathbb{Z}}\frac{\sigma}{\varepsilon^2}u_j^\varepsilon\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx$$

In this method the third term D_3 is more complicated to study

$$D_3 = \sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac12} - u^\varepsilon_{j-\frac12}}{\varepsilon} \left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} p^\varepsilon(x)dx\right) + \sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_{j+\frac12} - p^\varepsilon_{j-\frac12}}{\varepsilon} \left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)$$
$$+ \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2} \left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right).$$

It is decomposed in several pieces. We add and subtract in each fluxes the value of the unknowns in the cell. We also add and subtract to the two first integrals the value of the unknowns on the edge. Denoting by $\delta_j^\pm(g) = \frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} g(x)dx - g(x_{j\pm\frac12})$, one gets after rearrangements

$$\begin{aligned}
D_3 = &\sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac12} - u^\varepsilon_j}{\varepsilon}\delta_j^+(p^\varepsilon) + \sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_j - u^\varepsilon_{j-\frac12}}{\varepsilon}\delta_j^-(p^\varepsilon) \\
&+ \sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_{j+\frac12} - p^\varepsilon_j}{\varepsilon}\delta_j^+(u^\varepsilon) + \sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_j - p^\varepsilon_{j-\frac12}}{\varepsilon}\delta_j^-(u^\varepsilon) \\
&- \sum_{j\in\mathbb{Z}} \frac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon} p^\varepsilon_j - \sum_{j\in\mathbb{Z}} \frac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} u^\varepsilon_j \\
&+ \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2} \left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)
\end{aligned}$$

Using the fluxes' definition (16), one can eliminate the pressure fluxes. With a Young's inequality $ab \le \alpha a^2 + \frac{1}{4\alpha}b^2$ where $\alpha > 0$, one gets

$$\sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_{j+\frac12} - p^\varepsilon_j}{\varepsilon}\delta_j^+(u^\varepsilon) = \sum_{j\in\mathbb{Z}} \frac{1}{\varepsilon}(u^\varepsilon_j - u^\varepsilon_{j+\frac12})\delta_j^+(u^\varepsilon) - \frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j u^\varepsilon_{j+\frac12}\delta_j^+(u^\varepsilon)$$

$$\le \alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12} - u^\varepsilon_j)^2}{\varepsilon} + \left(\frac{1}{4\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2}\right)\sum_{j\in\mathbb{Z}} \delta_j^+(u^\varepsilon)^2 + \frac{\sigma}{8\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j^2 \left(u^\varepsilon_{j+\frac12}\right)^2.$$

Using this expression in D_3 and using again Young's inequality, one gets for arbitrary $\alpha > 0$

$$\begin{aligned}
D_3 \le\ &\alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12} - u^\varepsilon_j)^2}{\varepsilon} + \alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j-\frac12} - u^\varepsilon_j)^2}{\varepsilon} \\
&+ \sum_{j\in\mathbb{Z}} \left(\left(\frac{1}{2\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2}\right)\left(\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2\right) + \frac{\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2}{2\varepsilon\alpha} \right) \\
&+ \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac12})^2 + (u^\varepsilon_{j+\frac12})^2}{\varepsilon} + \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2}\frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right) \\
&- \sum_{j\in\mathbb{Z}} u^\varepsilon_j \frac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} - \sum_{j\in\mathbb{Z}} p^\varepsilon_j \frac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon}.
\end{aligned}$$

We now sum all bounds contributing to $\mathscr{E}'(t)$ and we get:

$$
\begin{aligned}
\mathscr{E}'(t) \le (-1+\alpha) \sum_{j\in\mathbb{Z}} & \frac{(u^\varepsilon_{j+\frac12}-u^\varepsilon_j)^2+(u^\varepsilon_{j-\frac12}-u^\varepsilon_j)^2}{\varepsilon} \\
&+ \sum_{j\in\mathbb{Z}} \left(\left(\frac{1}{2\alpha\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\left(\delta^+_j(u^\varepsilon)^2+\delta^-_j(u^\varepsilon)^2\right) + \frac{\delta^+_j(p^\varepsilon)^2+\delta^-_j(p^\varepsilon)^2}{2\varepsilon\alpha}\right) \\
&+ \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon} \\
&+ \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2}\frac{u^\varepsilon_{j+\frac12}+u^\varepsilon_{j-\frac12}}{2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right) - \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{2\varepsilon}\frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon} \\
&+ \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} u^\varepsilon_j \left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right) - \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} (u^\varepsilon)^2(x)dx\right).
\end{aligned}
$$

We now examine the sum of all terms in the two last lines of the RHS of the above inequality , which we denote S. One finds

$$
\begin{aligned}
S = -\sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{2\varepsilon^2} & \left[\left(u^\varepsilon_{j-\frac12} - \frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx \right)^2 + \left(u^\varepsilon_{j+\frac12} - \frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx \right)^2 \right] \\
&+ \frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}} \left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx \right)\left(u^\varepsilon_j - u^\varepsilon_{j+\frac12} + u^\varepsilon_j - u^\varepsilon_{j-\frac12} \right) \\
&\le \frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}} \left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx \right)\left(u^\varepsilon_j - u^\varepsilon_{j+\frac12} + u^\varepsilon_j - u^\varepsilon_{j-\frac12} \right).
\end{aligned}
$$

Using another Young's inequality, one has for all $\widehat{\alpha} > 0$

$$
S \le \frac{\sigma^2}{8\widehat{\alpha}\varepsilon^3}\sum_{j\in\mathbb{Z}} \left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx \right)^2 + \widehat{\alpha}\sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_j - u^\varepsilon_{j+\frac12})^2 + (u^\varepsilon_j - u^\varepsilon_{j-\frac12})^2}{\varepsilon}.
$$

For example by choosing $\alpha = \frac12$ and $\widehat{\alpha} = \frac12$, and coming back to $\mathscr{E}'(t)$ we get

$$
\begin{aligned}
\mathscr{E}'(t) \;\le\; & \sum_{j\in\mathbb{Z}} \left(\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\left(\delta^+_j(u^\varepsilon)^2+\delta^-_j(u^\varepsilon)^2\right) + \frac{1}{\varepsilon}\left(\delta^+_j(p^\varepsilon)^2+\delta^-_j(p^\varepsilon)^2\right)\right) \qquad (33)\\
&+ \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon} + \frac{\sigma^2}{4\varepsilon^3}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)^2. \qquad (34)
\end{aligned}
$$

To estimate the contributions on the first line we use the following fact: for any quantity q, one can use $q(x_{j-\frac12}) = q(x) + \int_x^{x_{j-\frac12}} \frac{d}{ds}q(s)ds$ and integrate this expression in the cell Δx_j; we get $\sum_{j\in\mathbb{Z}} \delta^\pm_j(q)^2 \le h\|q\|^2_{H^1(\mathbb{R})}$. Therefore the first terms on the right hand side of (33) can be estimated as

$$
\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\int_0^t \sum_{j\in\mathbb{Z}}\left(\delta^+_j(u^\varepsilon)^2+\delta^-_j(u^\varepsilon)^2\right)dt \le 2h\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\|u^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R}))}.
$$

Since $\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R}))} \le t\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}$ and also $\frac{\sigma}{\varepsilon^2}\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R}))} \le \|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}$ by (2) and (3), one gets that

$$
\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\int_0^t \sum_{j\in\mathbb{Z}}\left(\delta^+_j(u^\varepsilon)^2+\delta^-_j(u^\varepsilon)^2\right)dt \le 2h\left(\frac{t}{\varepsilon}+\frac{1}{2}\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}. \qquad (35)
$$

A similar and simpler formula for the next terms is

$$\frac{1}{\varepsilon}\int_0^t \sum_{j\in\mathbb{Z}} \left(\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2\right) \le 2\frac{ht}{\varepsilon}\|p^\varepsilon\|^2_{H^1(\mathbb{R})} \le 2\frac{ht}{\varepsilon}\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}. \tag{36}$$

Next, using the assumption (2.1) on the mesh and the estimate (27), one controls the next term by

$$\int_0^T \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac12})^2 + (u^\varepsilon_{j+\frac12})^2}{\varepsilon} \le \frac{h}{4C_{\mathcal{M}}}\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\mathbb{R})}. \tag{37}$$

Finally the last term in (34) can be bounded as

$$\frac{\sigma^2}{4\varepsilon^3}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)^2 \le \frac{\sigma^2}{4\varepsilon^3}h\|u^\varepsilon\|^2_{L^2(\mathbb{R})}$$

so that

$$\frac{\sigma^2}{4\varepsilon^3}\int_0^t\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)^2 \le \frac{\sigma^2}{4\varepsilon^3}h\|u^\varepsilon\|^2_{L^2([0,t]\times\mathbb{R})} \le \frac{\sigma}{4\varepsilon}h\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\mathbb{R})} \tag{38}$$

by means of the energy identity. We note that

$$\|\mathbf{V}^\varepsilon(0)\|_{H^p(\mathbb{R})} \le (1+\varepsilon/\sigma)\|p_0\|_{H^{p+1}(\mathbb{R})} \le (1+1/\sigma)\|p_0\|_{H^{p+1}(\mathbb{R})} \quad \forall p\in\mathbb{N}. \tag{39}$$

So using (35-38) we obtain for all time $t \le T$

$$\begin{aligned}\mathscr{E}(t) &\le \mathscr{E}(0) + \left(\frac{t}{\varepsilon}+\frac12+\frac{t}{\varepsilon}+\frac{1}{4C_{\mathcal{M}}}+\frac{\sigma}{4\varepsilon}\right)h\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}\\ &\le (1+1/\sigma)\left(C_{\mathcal{A}}^2h+\frac{2t}{\varepsilon}+\frac12+\frac{1}{4C_{\mathcal{M}}}+\frac{\sigma}{4\varepsilon}\right)h\|p_0\|^2_{H^2(\mathbb{R})}\end{aligned}$$

where the initialization stage is estimated using (21). One obtains after integration

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le \sqrt{T}\left(\sqrt{1+1/\sigma}\times\sqrt{C_{\mathcal{A}}^2h\varepsilon+2T+\frac{\varepsilon}{2}+\frac{\varepsilon}{4C_{\mathcal{M}}}+\frac{\sigma}{4}}\right)\sqrt{\frac{h}{\varepsilon}}\|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}.$$

The constant in parentheses is $\sqrt{T}\sqrt{1+1/\sigma}\sqrt{C_{\mathcal{A}}^2h\varepsilon C_{\mathcal{M}}+2TC_{\mathcal{M}}+\frac{\varepsilon}{2}C_{\mathcal{M}}+\frac{\varepsilon}{4}+\frac{\sigma}{4}C_{\mathcal{M}}}/\sqrt{C_{\mathcal{M}}} \le \frac{{}_\downarrow C}{\sqrt{C_{\mathcal{M}}}}$ with

$${}_\downarrow C = \sqrt{T}\sqrt{1+1/\sigma}\times\sqrt{C_{\mathcal{A}}^2+2T+\frac12+\frac14+\frac{\sigma}{4}}.$$

The proof is ended. □

2.5 Study of $\|P_h^0 - P^0\|$

We first recall a fundamental error estimate [14] for the diffusion limit scheme (18-19).

Lemma 2.7. *There exists a constant $C_\downarrow > 0$ independent of $h, \varepsilon, C_{\mathcal{M}}$, with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0;T]\times\mathbb{R})} \le \frac{C_\downarrow}{\sqrt{C_{\mathcal{M}}}}h\|p_0\|_{H^2(\mathbb{R})}. \tag{40}$$

Proof. We use a method that one can find in Eymard-Gallouet-Herbin [14]. It is based on a notion of consistency for finite volumes schemes. We set

$$s_j = \partial_{xx} p(x_j) - \frac{\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_{j-\frac{1}{2}})}{\Delta x_j} \text{ and } r_{j+\frac{1}{2}} = \partial_x p(x_{j+\frac{1}{2}}) - \frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}},$$

so that one has the identity

$$\frac{d}{dt} p(x_j) - \frac{1}{\sigma \Delta x_j} \left(\frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}} - \frac{p(x_j) - p(x_{j-1})}{\Delta x_{j-\frac{1}{2}}} \right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma \Delta x_j}.$$

We next introduce the difference $e_j = p(x_j) - p_j$ which satisfies

$$\frac{d}{dt} e_j - \frac{1}{\sigma \Delta x_j} \left(\frac{e_{j+1} - e_j}{\Delta x_{j+\frac{1}{2}}} - \frac{e_j - e_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma \Delta x_j}$$

with $e_j(0) = 0$ for all j. By multiplying this equation by e_j and denoting by $\|e_h\|^2_{L^2(\mathbb{R})} = \sum_j \Delta x_j e_j^2$, one finds that

$$\frac{1}{2} \frac{d}{dt} \|e_h\|^2_{L^2(\mathbb{R})} + \frac{1}{\sigma} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} = \frac{1}{\sigma} \sum_j \Delta x_j s_j e_j + \frac{1}{\sigma} \sum_j r_{j+\frac{1}{2}} (e_j - e_{j+1}).$$

The Cauchy-Schwarz inequality yields

$$\sum_j r_{j+\frac{1}{2}} (e_j - e_{j+1}) \leq \frac{1}{2} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} + \frac{1}{2} \sum_j \Delta x_{j+\frac{1}{2}} r^2_{j+\frac{1}{2}}.$$

One finds out with natural notations

$$\frac{1}{2} \frac{d}{dt} \|e_h\|^2_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} \leq \frac{1}{\sigma} \|s_h\|_{L^2(\mathbb{R})} \|e_h\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|r_h\|^2_{L^2(\mathbb{R})}. \tag{41}$$

Using the definitions of the truncation error s_h, one easily obtains by using classical arguments $\|s_h\|_{L^2([0,T]\times\mathbb{R})} \leq \sqrt{2} h \|\partial_{xxx} p\|_{L^2([0,T]\times\mathbb{R})}$: since p satisfies the diffusion equation (12), one gets $\|\partial_{xxx} p\|_{L^2([0,T]\times\mathbb{R})} \leq \sqrt{\sigma/2} \|\partial_{xx} p_0\|_{L^2(\mathbb{R})}$; one gets $\|s_h\|_{L^2([0,T]\times\mathbb{R})} \leq \sqrt{\sigma} h \|\partial_{xx} p_0\|_{L^2(\mathbb{R})}$. The same manipulations on the second truncation error r_h yield

$$\|s_h\|_{L^2([0,T]\times\mathbb{R})} + \|r_h\|_{L^2([0,T]\times\mathbb{R})} \leq \sqrt{\sigma} h \|p_0\|_{H^2(\mathbb{R})}. \tag{42}$$

One gets the bound from (41)

$$\|e_h\|^2_{L^2(\mathbb{R})}(t) \leq e_h\|^2_{L^2(\mathbb{R})}(0) + \int_0^t \frac{1}{\sigma} \|s_h\|_{L^2(\mathbb{R})} \|e_h\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|r_h\|^2_{L^2([0,T]\times\mathbb{R})}.$$

The use of the lemma (B.2), which is a corrolary of the Bihari's inequality, gives us:

$$\|e_h\|^2_{L^2([0,T]\times\Omega)} \leq \frac{1}{2} T \left(2 \sqrt{\|e_h\|^2_{L^2(\Omega)}(0) + \frac{1}{\sigma} \|r_h\|^2_{L^2([0,T]\times\Omega)}} + \frac{1}{\sigma} \sqrt{T} \|s_h\|_{L^2([0,T]\times\Omega)} \right)^2. \tag{43}$$

The initial value is bounded using (22) and taking into account (42) we obtain

$$\|e_h\|_{L^2([0,T]\times\Omega)} \leq \sqrt{\frac{T}{2}} \left(2 \sqrt{C_{\mathcal{A}} + \frac{1}{\sqrt{\sigma}}} + \sqrt{\frac{T}{\sigma}} \right) h \|p_0\| = B \; h \|p_0\|. \tag{44}$$

We also deduce from (41)

$$\int_0^T \sum_j \frac{(e_{j+1}-e_j)^2}{\Delta x_{j+\frac12}} \le \left(\|s_h\|_{L^2([0,T]\times\mathbb{R})} + \|r_h\|_{L^2([0,T]\times\mathbb{R})}\right)^2 + \|e_h\|^2_{L^2([0,T]\times\mathbb{R})}$$
$$\le \left(\sigma + B^2\right) h^2 \|p_0\|^2_{H^2(\mathbb{R})}$$

The other term that we must bound in (40) is $f_h = \varepsilon\left(v(x_j) - v_j\right) 1_j(x) = -\varepsilon\left(\frac{\partial_x p(x_j)}{\sigma} + v_j\right) 1_j(x)$ with v_j defined in (20). It yields

$$\|f_h\|_{L^2([0;T]\times\mathbb{R})} = \frac{\varepsilon}{2\sigma}\left(\int_0^T \sum_j \Delta x_j \left| \frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \partial_x p(x_j) + \frac{p_j - p_{j-1}}{\Delta x_{j-\frac12}} - \partial_x p(x_j)\right|^2\right)^{\frac12} \tag{45}$$

where the definition of e_j yields $\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \partial_x p(x_j) = \left(\partial_x p(x_{j+\frac12}) - \partial_x p(x_j)\right) + \left(\frac{e_{j+1}-e_j}{\Delta x_{j+\frac12}}\right) - r_{j+\frac12}$. One gets from the triangular inequality

$$\left(\int_0^T \sum_j \Delta x_{j+\frac12}\left(\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \partial_x p(x_j)\right)^2\right)^{\frac12} \le \left(\int_0^T \sum_j \Delta x_{j+\frac12}\left(\partial_x p(x_{j+\frac12}) - \partial_x p(x_j)\right)^2\right)^{\frac12}$$
$$+ \left[\sigma + B^2\right]^{\frac12} h\|p_0\|_{H^2(\mathbb{R})} + \|r_h\|_{L^2([0,T]\times\mathbb{R})}.$$

Since $\left(\int_0^T \sum_j \Delta x_{j+\frac12}\left(\partial_x p(x_{j+\frac12}) - \partial_x p(x_j)\right)^2\right)^{\frac12} \le h\sqrt{\frac{\sigma}{2}}\|p_0\|_{H^1(\mathbb{R})}$ and the estimate (42) holds, one gets

$$\left(\int_0^T \sum_j \Delta x_{j+\frac12}\left(\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \partial_x p(x_j)\right)^2\right)^{\frac12} \le \left(\left[\sigma + B^2\right]^{\frac12} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma}\right) h\|p_0\|_{H^2(\mathbb{R})}. \tag{46}$$

Taking into account that the weight Δx_j (45) is different from the weight $\Delta x_{j+\frac12}$ in (46), one gets

$$\|f_h\|_{L^2([0;T]\times\mathbb{R})} \le \frac{\varepsilon}{\sqrt{C_{\mathcal{M}}}}\left(\left[\sigma + B^2\right]^{\frac12} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma}\right) h\|p_0\|_{H^2(\mathbb{R})} \le \frac{1}{\sqrt{C_{\mathcal{M}}}}\left(\left[\sigma + B^2\right]^{\frac12} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma}\right) h\|p_0\|_{H^2(\mathbb{R})}, \tag{47}$$

since $\varepsilon \le 1$. Finally, the difference $\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|^2_{L^2([0;T]\times\mathbb{R})} = \|e_h\|^2_{L^2([0;T]\times\mathbb{R})} + \|f_h\|^2_{L^2([0;T]\times\mathbb{R})}$ is bounded using (44) and (47)

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0;T]\times\mathbb{R})} \le \frac{C_\downarrow}{\sqrt{C_{\mathcal{M}}}} h\|p_0\|_{H^2(\mathbb{R})}$$

and the constant $C_\downarrow$ with the definition of B by (44), has, at most, a linear growth in time. It ends the proof. □

2.6 Study of $\|P_h^\varepsilon - P_h^0\|$

In this section we prove an error estimate between the solution of the scheme (29) and the solution of the diffusion scheme (18). It is necessary to use some comparison estimates between the initial data of P_h^ε and P_h^0.

Lemma 2.8. *There exists a constant $C^{\rightarrow} > 0$ independent of $h, \varepsilon, C_{\mathcal{M}}$ and growth as $T^{\frac32}$ for large T such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le \frac{C^{\rightarrow}}{C_{\mathcal{M}}}\varepsilon\|p_0\|_{H^2(\mathbb{R})}. \tag{48}$$

Proof. For practical reasons we use the formulation (29) of the hyperbolic scheme which is equivalent to (15-16) and we reformulate the diffusion scheme (18-20) as

$$\begin{cases} \Delta x_j \dfrac{d}{dt} p_j + \dfrac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\varepsilon} = 0, \\ \\ \Delta x_j \dfrac{d}{dt} u_j - \dfrac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{\varepsilon} + \dfrac{2}{\varepsilon} u_j = \Delta x_j R_j, \\ \\ p_j - p_{j+1} + u_j + u_{j+1} = 2u_{j+\frac{1}{2}} + \sigma \Delta x_{j+\frac{1}{2}} \dfrac{u_{j+\frac{1}{2}}}{\varepsilon} + \Delta x_{j+\frac{1}{2}} S_{j+\frac{1}{2}}, \\ u_{j+\frac{1}{2}} = -\dfrac{\varepsilon}{\sigma} \dfrac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}}, \\ u_j = \dfrac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{2}, \end{cases} \tag{49}$$

where the error terms are R_j and $S_{j+\frac{1}{2}}$. A simple computation using the last two identities in (49) yields

$$R_j = \frac{d}{dt} u_j \text{ and } S_{j+\frac{1}{2}} = \frac{1}{\Delta x_{j+\frac{1}{2}}} \left(u_j + u_{j+1} - 2u_{j+\frac{1}{2}} \right).$$

One has from the triangular inequality applied to $u_j = \frac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{2}$

$$\left\| \frac{d}{dt} u_h \right\|_{L^2([0,T] \times \mathbb{R})} = \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_j \left| \frac{d}{dt} u_j \right|^2}$$

$$\leq \frac{1}{\sqrt{C_{\mathcal{M}}}} \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} u_{j+\frac{1}{2}} \right|^2} = \sqrt{\frac{\varepsilon}{C_{\mathcal{M}}}} \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} v_{j+\frac{1}{2}} \right|^2}.$$

Since the scheme is invariant with respect to the time variable, one can apply (31) to the derivative with respect to time. It yields

$$\sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} v_{j+\frac{1}{2}} \right|^2} \leq \sqrt{\frac{\sigma}{2}} \left\| \frac{d}{dt} p_h(0) \right\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2\sigma}} \|p_0\|_{H^2(\mathbb{R})}$$

where the last inequality is from the well preparedness of the initial data, as detailed in proposition 2.9. So one has the bound

$$\|R\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon}{\sqrt{2\sigma C_{\mathcal{M}}}} \|p_0\|_{H^2(\mathbb{R})}. \tag{50}$$

Using the definitions of u_j (49), $S_{j+\frac{1}{2}}$ can be written in terms of $\frac{d}{dt} p_j$ and $\frac{d}{dt} p_{j+1}$

$$S_{j+\frac{1}{2}} = \frac{\varepsilon}{2} \left(\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} \frac{d}{dt} p_j - \frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}} \frac{d}{dt} p_{j+1} \right).$$

Using the technical proposition 2.9, one finds out that $S = (S_{j+\frac{1}{2}})_{j \in \mathbb{Z}}$ satisfies

$$\|S\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon}{C_{\mathcal{M}}} \left\| \frac{d}{dt} p \right\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon \sqrt{T}}{\sigma C_{\mathcal{M}}} \|p_0\|_{H^2(\mathbb{R})}. \tag{51}$$

We now introduce the differences

$$e_j = p_j - p_j^\varepsilon, \quad f_j = u_j - u_j^\varepsilon \text{ and } f_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}^\varepsilon. \tag{52}$$

Let us look at the difference between the scheme (29) and (49). We get

$$\begin{cases} \Delta x_j \frac{d}{dt} e_j + \frac{f_{j+\frac12} - f_{j-\frac12}}{\varepsilon} = 0, \\ \Delta x_j \frac{d}{dt} f_j - \frac{f_{j+\frac12} + f_{j-\frac12}}{\varepsilon} + \frac{2}{\varepsilon} f_j = \Delta x_j R_j, \\ e_j - e_{j+1} + f_j + f_{j+1} - 2 f_{j+\frac12} - \sigma \Delta x_{j+\frac12} \frac{f_{j+\frac12}}{\varepsilon} = \Delta x_{j+\frac12} S_{j+\frac12}. \end{cases}$$

We use the notation $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 = \sum_j \Delta x_j (e_j^2 + f_j^2)$. Using the same kind of proof than for the L^2 stability of proposition 2.4, one gets that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \sum_j \Delta x_j R_j f_j - \sum_j \Delta x_{j+\frac12} \frac{f_{j+\frac12}}{\varepsilon} S_{j+\frac12} - \frac{\sigma}{\varepsilon^2} \sum_{j\in\mathbb{Z}} \Delta x_{j+\frac12} f_{j+\frac12}^2.$$

Using a Young's inequality on the second term of the right side of this inequality, one finds out that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \sum_j \Delta x_j R_j f_j + \frac{1}{4\sigma} \sum_j \Delta x_{j+\frac12} S_{j+\frac12}^2. \tag{53}$$

Using the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \|R\|_{L^2(\mathbb{R})} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma}\|S\|_{L^2(\mathbb{R})}^2.$$

Integrating in time on $[0,t]$

$$\|\mathbf{V}_h^\varepsilon(t) - \mathbf{W}_h^\varepsilon(t)\|_{L^2(\mathbb{R})}^2 \le \left(\int_0^t \|\mathbf{V}_h^\varepsilon(t) - \mathbf{W}_h^\varepsilon(t)\|_{L^2(\mathbb{R})} \|R\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma}\|S\|_{L^2([0,T]\times\mathbb{R})}^2 + \|\mathbf{V}_h^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 \right).$$

Another use of the Bihari's inequality, lemma (B.2), yields

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})}^2 \le \frac{1}{2} T \left(2\sqrt{\|\mathbf{V}_h^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 + \int_0^T \frac{1}{2\sigma}\|S\|_{L^2([0,T]\times\mathbb{R})}^2} + \sqrt{T}\|R\|_{L^2([0,T]\times\mathbb{R})}^2 \right)^2$$

Finally, using the previous estimates on R, S, the well-preparedness of the data (23) one gets

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})}^2 \le \frac{1}{2} T \left(2\sqrt{(C_{\mathcal{A}} h\varepsilon)^2 + \frac{\varepsilon^2 T^2}{4\sigma^3 C_{\mathcal{M}}^2}} + \sqrt{T} \frac{\varepsilon^2}{2C_{\mathcal{M}}} \right)^2 \|p_0\|_{H^2(\mathbb{R})}^2$$

The proof is ended. □

Proposition 2.9 (Technical result). *The bound* $\sqrt{\sum_j \Delta x_j (\frac{d}{dt} p_j)^2(t)} \le \sigma^{-1}\|p_0\|_{H^2(\mathbb{R})}$ *holds at any time.*

Proof. By linearity of the diffusion scheme, $z_h = \frac{d}{dt} p_h$ is solution of P_h^0:

$$\Delta x_j \frac{d}{dt} z_j - \frac{1}{\sigma}\left(\frac{z_{j+1} - z_j}{\Delta x_{j+\frac12}} - \frac{z_j - z_{j-1}}{\Delta x_{j-\frac12}} \right) = 0,$$

with initial condition

$$z_j(0) = \frac{d}{dt} p_0(x_j) = \frac{1}{\Delta x_j \sigma}\left(\frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac12}} - \frac{p_0(x_j) - p_0(x_{j-1})}{\Delta x_{j-\frac12}} \right). \tag{54}$$

One gets from a Taylor expansion with integral residue that

$$\left| \frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j) \right| \leq \int_{x_j}^{x_{j+1}} |\partial_{xx} p_0(y)|\, dy.$$

Similarly one has the bound $\left| \frac{p_0(x_j)-p_0(x_{j-1})}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j) \right| \leq \int_{x_{j-1}}^{x_j} |\partial_{xx} p_0(y)|\, dy$. Therefore $|z_j(0)| \leq \frac{1}{\Delta x_j \sigma} \int_{x_{j-1}}^{x_{j+1}} |\partial_{xx} p_0(y)|\, dy$ from which the bound $\sqrt{\sum_j \Delta x_j z_j^2(0)} \leq \sigma^{-1} \|p_0\|_{H^2(\mathbb{R})}$ is deduced. Since the scheme P_h^0 is stable in L^2, this bound is true at any time. Considering (54) the discrete second derivative attached to P_h^0 is bounded at any time, which ends the proof of the claim. □

2.7 End of the proof of uniform AP property

Theorem 2.10. *Assuming a sufficiently smooth well prepared initial data, the scheme P_h^ε converges to P^ε at order at least $\frac{1}{3}$ in $L^2([0,T] \times \mathbb{R})$, uniformly with respect to ε*

Proof. All the previous estimates show that (9-8) are true with $a = 1$, $b = c = \frac{1}{2}$ and $d = 1$. More specifically, estimates (32), (48), (40) and (24) shows that

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}\,,\; h + 2\varepsilon \right) \|p_0\|_{H^3(\mathbb{R})}$$

where

$$C = \max\left[\frac{{}_{\downarrow}C}{\sqrt{C_{\mathcal{M}}}}, \frac{C^{\rightarrow}}{C_{\mathcal{M}}}, \frac{C_{\downarrow}}{\sqrt{C_{\mathcal{M}}}}, C_{\leftarrow} \right]$$

and behaves less than $T^{\frac{3}{2}}$ for large T. Using the general method described at the beginning of this work in proposition 1.3, one obtains the convergence estimate $\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \leq C(T) h^q$ with the order of convergence $q = \frac{ac}{a+b} = \frac{1}{3}$.

□

3 The 2D case

In this section we prove the uniform convergence of the solution of the diffusion AP scheme introduced in [7] to the solution of the hyperbolic heat equation. The structure of our proof is globally the same as in the previous section. However two major difficulties will be treated: a) the first one consists in the adaptation to our problem of a combination of specific finite volumes techniques for hyperbolic and parabolic equations; b) the second one is to derive new bounds for the scheme $\mathbf{DA}_h^\varepsilon$.

The model problem is the hyperbolic heat equation in the domain $\Omega =]0,1[^2$ with periodic boundary conditions and well-prepared data

$$\mathbf{P}^\varepsilon : \quad \begin{cases} \partial_t p^\varepsilon + \dfrac{1}{\varepsilon} \operatorname{div}(\mathbf{u}^\varepsilon) = 0, \\[2ex] \partial_t \mathbf{u}^\varepsilon + \dfrac{1}{\varepsilon} \nabla p^\varepsilon = -\dfrac{\sigma}{\varepsilon^2} \mathbf{u}^\varepsilon, \\[2ex] p^\varepsilon(t=0) = p_0,\; \mathbf{u}^\varepsilon(t=0) = \mathbf{u}_0^\varepsilon = -\frac{\varepsilon}{\sigma} \nabla p_0. \end{cases}$$

When ε tends to zero, this problem admits the following diffusion limit

$$\mathbf{P}^0 : \qquad \partial_t p - \frac{1}{\sigma} \operatorname{div}(\nabla p) = 0, \qquad p(t=0) = p_0.$$

The rescaled gradient is $\mathbf{v} = -\frac{1}{\sigma} \nabla p$. We will admit the following proposition, the proof of which can be easily obtained by a method similar to the one of proposition 2.3.

Proposition 3.1. *The error between the two solutions can be upper bounded by*

$$\|p^\epsilon - p\|_{L^\infty([0,T];H^n(\Omega))} + \|\mathbf{v}\|_{L^\infty([0,T];H^n(\Omega))} \le \frac{T}{\sigma^2}\varepsilon \|p_0\|_{H^{3+n}(\Omega)}, \qquad n \in \mathbb{N}. \tag{55}$$

Proof. The structure of the proof in the $L^\infty([0,T];L^2(\Omega))$ norm is the same as the one of proposition 2.3. Since the coefficients of the problem are constant, similar bounds are obtained at any order of derivation which proves the estimate for any $n > 0$. □

3.1 Definition of $\mathbf{P}_h^\varepsilon$

Let us consider an unstructured mesh in dimension 2. The mesh is defined by a finite number of vertices $\mathbf{x}_r$ and cells Ω_j. We denote $\mathbf{x}_j$ a point chosen arbitrarily inside Ω_j. For simplicity we will call this point the center of the cell. By convention the vertices are listed counter-clockwise $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_{r+1}$ with coordinates $\mathbf{x}_r = (x_r, y_r)$. We note $l_{jr}\mathbf{n}_{jr}$ the vector as follows

$$l_{jr} = \frac{1}{2}\text{dist}\left(\mathbf{x}_{r-1}, \mathbf{x}_{r+1}\right) \text{ and } \mathbf{n}_{jr} = \frac{1}{2l_{jr}}\left(\mathbf{x}_{r+1} - \mathbf{x}_{r-1}\right)^\perp. \tag{56}$$

This notion of a corner vector can be rigorously introduced also in any dimension using the definition [13]. The scalar product of two vectors is denoted as $(\mathbf{x}, \mathbf{y})$.

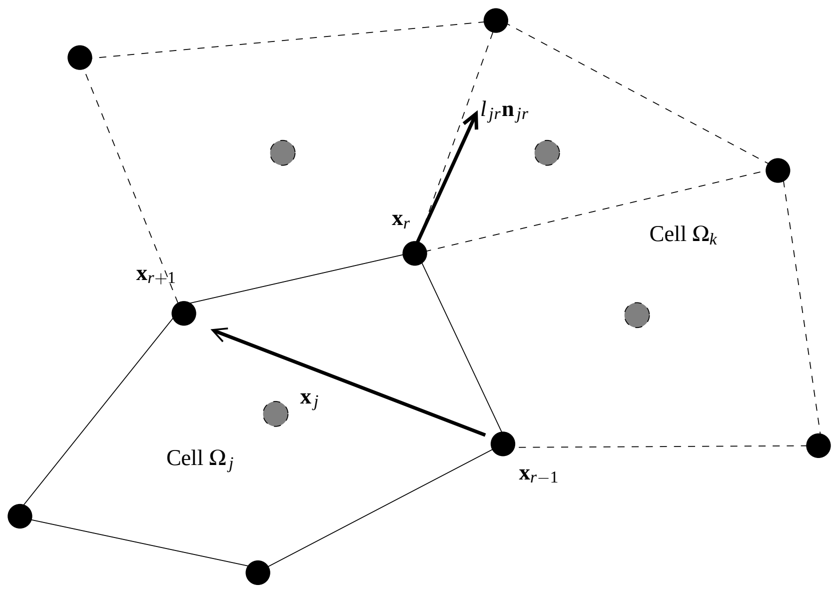

Figure 4: Notation for node formulation. The corner length l_{jr} and the corner normal $\mathbf{n}_{jr}$ are defined in equation (56). The point $\mathbf{x}_j$ is an arbitrary point inside the cell, typically the centroid of the cell or an averaged of the corners.

The numerical approximation of the problem $\mathbf{P}^\varepsilon$ that we study is the JL-(b) scheme defined in [7]

$$\mathbf{P}_h^\varepsilon : \quad \begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} p_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \mid \Omega_j \mid \dfrac{d}{dt}\mathbf{u}_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}\mathbf{n}_{jr} p_{jr}^\varepsilon = -\dfrac{\sigma}{\varepsilon^2}\displaystyle\sum_r \widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \end{cases} \tag{57}$$

with for simplicity point wise initial data $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$ and $\mathbf{u}_j^\varepsilon(0) = -\varepsilon\sigma^{-1}\nabla p_0(\mathbf{x}_j)$. The fluxes are defined by the so-called corner problem

$$\left\{\begin{array}{l} p_{jr}^\varepsilon - p_j^\varepsilon = (\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \dfrac{\sigma}{\varepsilon}(\mathbf{x}_r - \mathbf{x}_j, \mathbf{u}_r^\varepsilon), \\ \sum_j l_{jr} p_{jr}^\varepsilon \mathbf{n}_{jr} = 0. \end{array}\right. \tag{58}$$

This corner problem has been introduced in [7] as a multidimensional version of the 1D Jin-Levermore technique [23]. Its solution is provided by the solution of the linear system

$$\left(\sum_j \widehat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\right)\mathbf{u}_r^\varepsilon = \sum_j l_{jr} p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j^\varepsilon, \tag{59}$$

where the geometry of the mesh is used to define the matrices $\widehat{\alpha}_{jr}$ and $\widehat{\beta}_{jr}$

$$\widehat{\alpha}_{jr} = l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr}, \text{ and } \widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j). \tag{60}$$

We will use the notations $A_j = \sum_r \widehat{\alpha}_{jr}$, $A_r = \sum_j \widehat{\alpha}_{jr}$ and $B_r = \sum_j \widehat{\beta}_{jr}$. By comparison with the scheme P_h^ε in dimension one, one sees that the multi-dimensional scheme (57-60) is more tricky than the 1D scheme (15-16).

Starting from (57) and taking into account of the definitions of the fluxes (58) and also the identity $\sum_r l_{jr}\mathbf{n}_{jr} = 0$, the scheme $\mathbf{P}_h^\varepsilon$ can also be rewritten as

$$\mathbf{P}_h^\varepsilon : \quad \left\{\begin{array}{l} \mid \Omega_j \mid \dfrac{d}{dt}p_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \mid \Omega_j \mid \dfrac{d}{dt}\mathbf{u}_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon)\mathbf{n}_{jr} = 0 \end{array}\right. \tag{61}$$

When $\varepsilon \to 0$ the scheme $\mathbf{P}_h^\varepsilon$, see (57) or (61), admits the limit diffusion scheme $\mathbf{P}_h^0$

$$\mathbf{P}_h^0 : \quad \left\{\begin{array}{l} |\Omega_j| \dfrac{d}{dt}p_j + \displaystyle\sum_r l_{jr}\Big(\mathbf{v}_r, \mathbf{n}_{jr}\Big) = 0, \\ \mathbf{v}_r = \dfrac{1}{\sigma}B_r^{-1}\displaystyle\sum_j l_{jr} p_j \mathbf{n}_{jr}, \end{array}\right. \tag{62}$$

with $B_r = \sum_j l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)$. We define additionally $\mathbf{v}_j$ by a kind of mean

$$\left(\sum_r \widehat{\alpha}_{jr}\right)\mathbf{v}_j = \sum_r \widehat{\alpha}_{jr}\mathbf{v}_r.$$

This is well defined since the matrix $\sum_r \widehat{\alpha}_{jr}$ is symmetric positive by definition of the $\widehat{\alpha}_{jr}$.

3.2 Definition of $\mathbf{DA}_h^\varepsilon$

We define now that is call thereafter the "diffusion approximation" scheme. We just neglect the time derivative in the second equation, that we make $\partial_t \mathbf{u}_j^\varepsilon = 0$ for (61). It leads to the scheme

$$\mathbf{DA}_h^\varepsilon : \quad \left\{\begin{array}{l} \mid \Omega_j \mid \dfrac{d}{dt}p_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r (l_{jr}\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \frac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon)\mathbf{n}_{jr} = 0 \\ \left(\displaystyle\sum_j \widehat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\right)\mathbf{u}_r^\varepsilon = \displaystyle\sum_j l_{jr} p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j^\varepsilon \end{array}\right. \tag{63}$$

This scheme depends of two parameters, the size of the mesh h and the small parameter ε. We notice that $\mathbf{DA}_h^\varepsilon \neq \mathbf{P}_h^0$ for $\varepsilon > 0$, and that $\lim_{\varepsilon\to 0^+}\mathbf{DA}_h^\varepsilon = \mathbf{P}_h^0$. The point wise initial data for (63) is $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$. There is no need of initial data for $(u_j^\varepsilon(0))$, which will be obtained as a function of $(p_j^\varepsilon(0))$ by solving a linear system.

3.3 Mesh assumptions

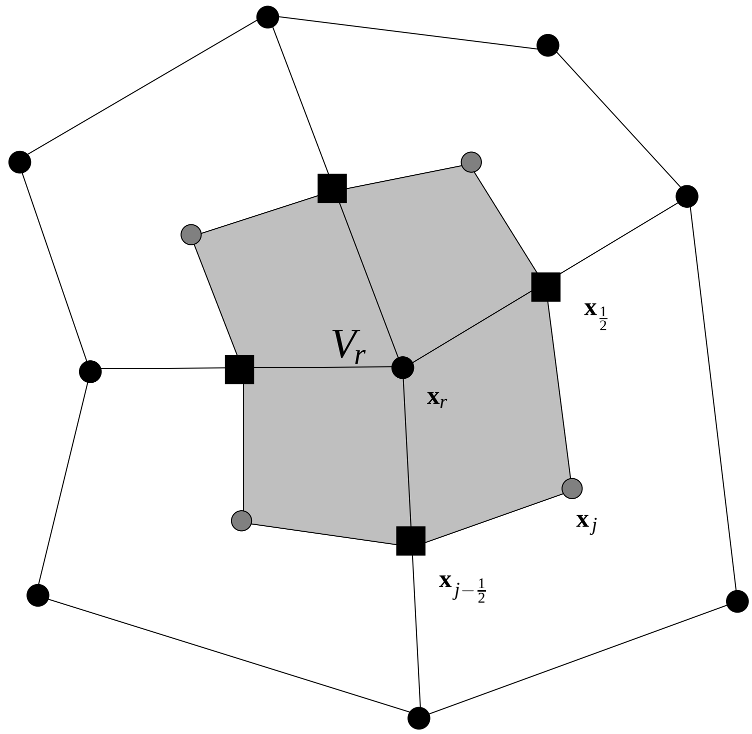

Figure 5: Definition of the control volume V_r around vertex $\mathbf{x}_r$. The control volume around the vertex $\mathbf{x}_r$ is defined by the closed loop that joins the center of the cells ($\mathbf{x}_j$'s) and the middle of the edges ($\mathbf{x}_{j+\frac{1}{2}}$'s).

The characteristic length of the mesh is $h = \max_j (\text{diam}(\Omega_j))$, so that

$$\begin{cases} l_{jr} \le h, & \forall j, r, \\ |\Omega_j| \le h^2, & \forall j. \end{cases} \tag{64}$$

The control volume V_r around the vertex $\mathbf{x}_r$ is defined by the closed loop $\ldots, \mathbf{x}_{j-\frac{1}{2}}, \mathbf{x}_j, \mathbf{x}_{j+\frac{1}{2}}, \ldots$. Here the $\mathbf{x}_j$'s are the center of the cells, and the $\mathbf{x}_{j+\frac{1}{2}}$'s are the middle of the edges around the vertices $\mathbf{x}_r$. A typical example is depicted in figure 5.

Additional geometrical assumptions are always necessary in dimension greater than one to guarantee some minimal regularity of the mesh. We make the usual assumptions listed below from 1 to 3. The last items are more specific.

Hypothesis 3.2. *Our geometrical assumptions will be the following*

1. *The numbers of cells which share a node r is bounded independently of h, which means there exists $P \in \mathbb{N}$ independent of h such that*

$$\sum_j \delta_{jr} \le P. \tag{65}$$

   For example, for a structured mesh of quadrangular cells $P = 4$.

2. *For each cell of the mesh, the number of edges is bounded independently of h, or equivalently the numbers of vertices for a cell is bounded independently of h.*

3. *The mesh is regular in the sense that there exists a universal constant* $C_{\mathcal{M}} > 0$ *such that the inverse inequalities hold:*

$$C_{\mathcal{M}} h \le l_{jr}, \quad \forall j, r \qquad \textit{uniformly with respect to } h \tag{66}$$

where $\mathbf{x}_r$ *is a vertex of the cell* Ω_j*, and*

$$C_{\mathcal{M}} h^2 \le |\Omega_j|\,, \quad \forall j \qquad \textit{uniformly with respect to } h. \tag{67}$$

and

$$C_{\mathcal{M}} h^2 \le |V_r| \le P h^2, \quad \forall r \qquad \textit{uniformly with respect to } h. \tag{68}$$

We recall that V_r *is the volume control (centered on* $\mathbf{x}_r$*) and* Ω_j *is the cell* j*. The inequality* $|V_r| \le P h^2$ *is immediate to check on the figure 5.*

4. *A consequence of the items 1-3 is that there exists a constant* $\alpha > 0$ *such that*

$$(A_j \mathbf{u}, \mathbf{u}) \ge \alpha h (\mathbf{u}, \mathbf{u}), \quad A_j = \sum_r \widehat{\alpha}_{jr}. \tag{69}$$

It can be proved with a geometrical identity that we borrow from [13] (proposition 8).

5. *The matrix* $B_r = \sum_j \widehat{\beta}_{jr}$ *is positive in the sense that*

$$(B_r \mathbf{u}, \mathbf{u}) = (B_r^s \mathbf{u}, \mathbf{u}) \ge \alpha |V_r| (\mathbf{u}, \mathbf{u}), \tag{70}$$

where $B_r^s = \frac{1}{2}(B_r + B_r^t)$ *is the symmetric part of* B_r*, and* α *is the same constant as in (69). Square meshes satisfy (70). This assumption is however not trivial to check in the general case. We point out [7] where sufficient conditions such that (70) is satisfied can be found; in particular it is shown that triangular meshes with all angles greater than* 12 *degrees satisfy it.*

3.4 Norms and error measurements

The quadratic norms below are usual integral norms. It yields for any cell centered quantity $f = (f_j)_{j \in \text{Cells}}$: $\|f\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j||f_j|^2}$. For vertex based quantity $g = (g_r)_{r \in \text{Vertices}}$, we use $\|g\|_{L^2(\Omega)} = \sqrt{\sum_r |V_r||g_r|^2}$: it is more a convention. Useful quantities are

- $\mathbf{V}_h^\varepsilon(t, \mathbf{x}) = \left(\sum_{j \in \text{Cells}} p_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}), \sum_{j \in \text{Cells}} \mathbf{u}_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}) \right)$ which is the solution of $\mathbf{P}_h^\varepsilon$.
- $\mathbf{V}^\varepsilon(t, \mathbf{x}) = (p^\varepsilon, \mathbf{u}^\varepsilon)(t, \mathbf{x})$ which is the solution of $\mathbf{P}^\varepsilon$,
- $\mathbf{W}_h^\varepsilon(t, \mathbf{x}) = \left(\sum_{j \in \text{Cells}} p_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}), \sum_{j \in \text{Cells}} \mathbf{u}_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}) \right)$ which is the solution of $\mathbf{DA}_h^\varepsilon$. Notice that an abuse of notations is made with the solution of $\mathbf{P}_h^\varepsilon$.
- $\mathbf{W}^\varepsilon(t, \mathbf{x}) = \left(p, -\frac{\varepsilon}{\sigma} \nabla p \right)(t, \mathbf{x})$ which is the solution of $\mathbf{P}^0$.

As in dimension one, the differences between these quantities are characterized at the initial time with a universal constant $C_{\mathcal{A}} > 0$ which indicates it can be related to the approximation/interpolation/projection of a smooth function on the mesh. We will use for example some bounds that can be obtained as by-product or corollary of the first technical inequality below.

$$\begin{cases} \|\mathbf{V}^\varepsilon(0) - \mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)} \le C_{\mathcal{A}} h \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)} \le C_{\mathcal{A}} h \|\mathbf{V}^1(0)\|_{H^2(\Omega)}, \\ \|\mathbf{W}^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\Omega)} \le C_{\mathcal{A}} h \|\mathbf{W}^\varepsilon(0)\|_{H^2(\Omega)}. \end{cases} \tag{71}$$

We will need additional technical estimates for the corner based Finite Volume scheme $\mathbf{P}_h^\varepsilon$. These technical estimates can be formulated as follows. Let f be a regular function. We define

$\delta_{j,r}(f) = \frac{1}{|\Omega_j|}\int_{\Omega_j} f d\mathbf{x} - f(\mathbf{x}_r)$ which is the interpolation error term that compares mean value in a cell Ω_j and point values at a vertex $\mathbf{x}_r$ of the same cell. Let $\Gamma_{j,r} = [\mathbf{x}_r, \mathbf{x}_{r+1}]$ be the edge oriented toward the outside of the cell j, with length $|\Gamma_{j,r}|$. We define also $\tilde{\delta}_{j,r}(h) = \frac{1}{|\Gamma_{j,r}|}\int_{\Gamma_{j,r}} h ds - \frac{h(\mathbf{x}_r)+h(\mathbf{x}_{r+1})}{2}$ which is another interpolation error contribution that compares the mean value and the mid sum, on the edge.

Proposition 3.3. *One has the technical inequalities*

$$|\delta_{jr}(f)| \leq C_{\mathcal{A}} \|f\|_{H^2(\Omega_j)} \tag{72}$$

and

$$|\widetilde{\delta}_{jr}(f)| \leq C_{\mathcal{A}} h \|f\|_{H^3(\Omega_j)} \tag{73}$$

Proof. These non optimal inequalities are consequences of classical approximations results. We will not prove them. However one can notice that the scaling is correct. That if a function f has its third derivatives bounded in $L^\infty(\Omega_j)$, then $\|f\|_{H^2(\Omega_j)} = O(h)$ because the problem is 2D: this is compatible with the fact that δ_{jr} is a first order difference. Similarly $h\|f\|_{H^3(\Omega_j)} = O(h^2)$ is compatible with the fact that $\widetilde{\delta}_{jr}$ is a second order difference. An alternative proof is by assuming that f is in $H^p(\Omega)$ for a sufficiently large p. Then by the Sobolev embeddings, all derivatives up to fourth order are in L^∞ which is enough to prove that (72) is a first order interpolation error term, and that (73) is a second order interpolation error term. In this case it also explains very simply why the constant $C_{\mathcal{A}}$ is independent of the mesh size. □

The first technical inequality is actually true for any points in the cell. So it allows to compare the mean value and the point value in the cell. This is why it yields (71) after summation over all cells and redefinition of $C_{\mathcal{A}}$.

As in dimension one, we will use constants ${}_{\downarrow}C$, $C^{\rightarrow}$, $C_{\downarrow}$ and $C_{\leftarrow}$ in the errors bounds for the four branches of the new AP diagram. The important point is that these constants are independent of h and ε. They have of course some dependence with respect to other parameters such as the constant of the mesh $C_{\mathcal{M}}$ for example, but we will not keep track of these dependence in order to simplify the notations. Nevertheless the interested reader can compare with the same estimates in dimension one where the dependence with respect to the mesh constant is indicated. A first result is the inequality (55) which yields the basic estimate for the lower branch of the AP diagram. It can be formalized as follows.

Lemma 3.4. *One has the estimate*

$$\|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C_{\leftarrow} \varepsilon \|p_0\|_{H^4(\Omega)} \tag{74}$$

where the constant $C_{\leftarrow}$ is independent of h and ε, with a growth in time less than $T^{\frac{3}{2}}$ by comparison with (55).

3.5 Study of $\|\mathbf{P}_h^\varepsilon - \mathbf{P}^\varepsilon\|_{\text{naive}}$

In this part, we exploit the hyperbolic nature of both $\mathbf{P}^\varepsilon$ and $\mathbf{P}_h^\varepsilon$ to obtain the main bound. As one will see below, the convergence estimate (75) is not trivial. It indicates that, for a problem with $O(\varepsilon^{-2})$ terms, a scheme converges, with h, with rate $O(\varepsilon^{-\frac{1}{2}})$ with respect to ε.

Lemma 3.5 (Naive estimate)**.** *There exists a constant ${}_{\downarrow}C$ independent of h and ε, with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^\infty([0,T]:L^2(\Omega))} \leq {}_{\downarrow}C \sqrt{\frac{h}{\varepsilon}} \|p_0\|_{H^4(\Omega)}. \tag{75}$$

The norm is slightly stronger than the $L^2([0,T]\times\Omega)$ needed to complete the proof.

3.5.1 Stability

We first prove the L^2 stability of the scheme P_h^ε defined in (57,58).

Proposition 3.6 (Stability)**.** *The semi-discrete general JL-(b) scheme defined by (57,58) is stable in the L^2 norm in the sense that $\frac{d}{dt}||\mathbf{V}_h^\varepsilon(t)|| \le 0$. Moreover we have the bounds*

$$\frac{\sigma}{\varepsilon^2}||\mathbf{u}_r^\varepsilon||_{L^2([0,T]\times\Omega)} \le \frac{1}{\alpha}||\mathbf{V}_h^\varepsilon(0)||_{L^2(\Omega)}, \tag{76}$$

$$\int_0^T \sum_j \sum_r l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 dt \le \varepsilon||\mathbf{V}_h^\varepsilon(0)||_{L^2(\Omega)}^2. \tag{77}$$

Proof. We define the functions p_h^ε and $\mathbf{u}_h^\varepsilon$ by $p_h^\varepsilon = p_j$ and $\mathbf{u}_h^\varepsilon = \mathbf{u}_j$ on Ω_j. We set for convenience $E(t) = ||\mathbf{V}_h^\varepsilon(t)||^2$. One has

$$E^{'}(t) = \frac{1}{2}\int_\Omega \frac{d}{dt}(|\, p_h^\varepsilon\,|^2 + (\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon)) = \int_\Omega p_h^\varepsilon \frac{d}{dt}p_h^\varepsilon + (\mathbf{u}_h^\varepsilon, \frac{d}{dt}\mathbf{u}_h^\varepsilon) = \sum_j |\Omega_j|\, p_j^\varepsilon \frac{d}{dt}p_j^\varepsilon + (\mathbf{u}_j^\varepsilon, \frac{d}{dt}\mathbf{u}_j^\varepsilon).$$

Using the definition of scheme

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \frac{1}{\varepsilon}\sum_j\sum_r (l_{jr}p_{j,r}^\varepsilon\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \tag{78}$$

Using (58) we expand the second term of the previous equation

$$\sum_j\sum_r (l_{jr}p_{j,r}^\varepsilon\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) = \sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_j^\varepsilon, \mathbf{n}_{jr}) + \sum_j\sum_r(\widehat{\alpha}_{jr}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon}\sum_j\sum_r(\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \tag{79}$$

Since $\sum_r l_{jr}\mathbf{n}_{jr} = 0$ the first term of (79) is zero. Summing on r the second equation of (58) and permuting the sums, we show that $0 = \sum_j\sum_r l_{jr}p_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})$ which yields that

$$0 = \sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \sum_j\sum_r((\widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr})\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j\sum_r(\widehat{\alpha}_{jr}\mathbf{u}_j^\varepsilon, \mathbf{u}_r^\varepsilon). \tag{80}$$

Plugging (79) and (80) in (78) and permuting the sums in $E'(t)$ gives

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r(\widehat{\alpha}_{jr}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \frac{\sigma}{\varepsilon^2}\sum_r\sum_j(\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$$

which gives

$$E^{'}(t) + \frac{1}{\varepsilon}\sum_r\sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \frac{\sigma}{\varepsilon^2}\sum_r(B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = 0. \tag{81}$$

By geometrical assumption (70) we have $E^{'}(t) \le 0$, that is the L^2 stability, and by integrating this equality on $[0, T]$ we obtain

$$E(T) + \int_0^T \frac{1}{\varepsilon}\sum_r\sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \int_0^T \frac{\sigma}{\varepsilon^2}\sum_r(B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = E(0)$$

Using again the geometrical assumption (70) for the terms $(B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$ we have

$$E(T) + \int_0^T \frac{1}{\varepsilon}\sum_r\sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \alpha\int_0^T \frac{\sigma}{\varepsilon^2}\sum_r |V_r|\; |\mathbf{u}_r^\varepsilon|^2 \le E(0)$$

which gives (76) and (77). The proof is ended. $\square$

3.5.2 Main estimate

Our goal now is to prove the lemma 3.5 as the consequence of propositions 3.7 to A.3. This part is the more technical one of the paper, but is essential to be able to use the general strategy of proposition 1.3 with convenient exponents. Like in 1D, we use the method introduced by Mazeran [29] and decompose the proof in several steps. We introduce $\mathscr{E}(t) = \frac{1}{2}\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|^2_{L^2(\Omega)}$. As for the 1D proof and for the sake of simplicity, for any quantity q, q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$.

Proposition 3.7. *One has the formula*

$$\mathscr{E}'(t) = -\frac{1}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + E_1 + E_2 + E_3 \tag{82}$$

where

$$\begin{aligned}
E_1 =& \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}\mathbf{n}_{j,r}(p_{jr}^\varepsilon - p_j^\varepsilon)\ ,\ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big),\\
E_2 =& \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}|\Big(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big)\\
E_3 =& \frac{\sigma}{\varepsilon^2}\sum_r\sum_j \left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\\
&- \frac{\sigma}{\varepsilon^2}\sum_j\int_{\Omega_j}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon)d\mathbf{x} - \frac{\sigma}{\varepsilon^2}\sum_r (B_r\mathbf{u}_r^\varepsilon, \boldsymbol{u}_r^\varepsilon).
\end{aligned}$$

Proof. We first consider the time derivative

$$\begin{aligned}
\mathscr{E}'(t) =& \underbrace{\int_\Omega (p_h^\varepsilon(p_h^\varepsilon)' + (\mathbf{u}_h^\varepsilon, (\mathbf{u}_h^\varepsilon)'))d\mathbf{x}}_{D_1} + \underbrace{\int_\Omega (p^\varepsilon(p^\varepsilon)' + (\mathbf{u}^\varepsilon, (\mathbf{u}^\varepsilon)'))d\mathbf{x}}_{D_2}\\
&+ \underbrace{\int_\Omega (-(p_h^\varepsilon)'p^\varepsilon - ((\mathbf{u}_h^\varepsilon)', \mathbf{u}^\varepsilon))d\mathbf{x}}_{D_3} + \underbrace{\int_\Omega (-p_h^\varepsilon(p^\varepsilon)' - (\mathbf{u}_h^\varepsilon, (\mathbf{u}^\varepsilon)'))d\mathbf{x}}_{D_4}.
\end{aligned}$$

One has thanks to (81)

$$D_1 = -\frac{1}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2}\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon).$$

One also directly has

$$D_2 = -\frac{\sigma}{\varepsilon^2}\int_\Omega (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon)d\mathbf{x} = -\frac{\sigma}{\varepsilon^2}\sum_j\int_{\Omega_j}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon)d\mathbf{x}.$$

Then, using the definition (57,58) of the scheme we have

$$\begin{aligned}
D_3 =& \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}\mathbf{u}_r^\varepsilon, \mathbf{n}_{j,r}\Big)\frac{1}{|\Omega_j|}\int_{\Omega_j} p^\varepsilon dx\\
&+ \frac{1}{\varepsilon}\sum_j \left(\sum_r l_{jr}\mathbf{n}_{jr}p_{j,r}^\varepsilon + \frac{\sigma}{\varepsilon}\sum_r \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)
\end{aligned}$$

Since $\sum_r l_{jr}\mathbf{n}_{jr}=0$, we can write

$$\begin{aligned}
D_3 &= \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon-\mathbf{u}_j^\varepsilon),\mathbf{n}_{j,r}\Big)\frac{1}{|\Omega_j|}\int_{\Omega_j} p^\varepsilon dx\\
&+\frac{1}{\varepsilon}\sum_j\Big(\sum_r l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon-p_j^\varepsilon)\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big)\\
&+\frac{\sigma}{\varepsilon^2}\Big(\sum_r\sum_j\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big).
\end{aligned}$$

One gets

$$\begin{aligned}
D_3 &= \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon-\mathbf{u}_j^\varepsilon),\mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon)+\frac{1}{\varepsilon}\sum_j\sum_r\Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon-p_j^\varepsilon)\ ,\ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big)\\
&+\frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon-\mathbf{u}_j^\varepsilon),\mathbf{n}_{j,r}\Big)p^\varepsilon(\mathbf{x}_r)+\frac{1}{\varepsilon}\sum_j\sum_r\Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon-p_j^\varepsilon)\ ,\ \mathbf{u}^\varepsilon(\mathbf{x}_r)\Big)\\
&+\frac{\sigma}{\varepsilon^2}\Big(\sum_r\sum_j\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big).
\end{aligned}$$

We have the identities $\sum_{j,r} l_{jr}\mathbf{n}_{jr}=0$ and $\sum_j l_{jr}\mathbf{n}_{jr}p_{j,r}^\varepsilon=0$ by definition (58). Therefore one can simplify the third and fourth term in the previous expression and get

$$\begin{aligned}
D_3 &= \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon-\mathbf{u}_j^\varepsilon),\mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon)+\frac{1}{\varepsilon}\sum_j\sum_r\Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon-p_j^\varepsilon)\ ,\ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big)\\
&-\frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}\mathbf{u}_j^\varepsilon,\mathbf{n}_{j,r}\Big)p^\varepsilon(\mathbf{x}_r)-\frac{1}{\varepsilon}\sum_j\sum_r\Big(l_{j,r}p_j^\varepsilon\mathbf{n}_{j,r}\ ,\ \mathbf{u}^\varepsilon(\mathbf{x}_r)\Big)\\
&+\frac{\sigma}{\varepsilon^2}\Big(\sum_r\sum_j\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big).
\end{aligned}$$

We now look at D_4. By definition, one has

$$D_4=\frac{1}{\varepsilon}\sum_j p_j^\varepsilon\sum_r\int_{\Gamma_{j,r}}(\mathbf{u}^\varepsilon,\tilde{\mathbf{n}}_{j,r})d\sigma+\frac{1}{\varepsilon}\sum_j\Big(\mathbf{u}_j^\varepsilon,\Big(\sum_r\int_{\Gamma_{j,r}}p^\varepsilon\tilde{\mathbf{n}}_{j,r}d\sigma+\frac{\sigma}{\varepsilon}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big)\Big)$$

where $\tilde{\mathbf{n}}_{j,r}$ is the normal to the edge $\Gamma_{j,r}=[\mathbf{x}_r,\mathbf{x}_{r+1}]$ oriented toward the outside of the cell j. This expression needs an important manipulation which is to approximate the integral on edges by corner values. This necessary manipulation is one of the ideas that was introduced in [29] in order to proceed to the numerical analysis of such corner based finite volume schemes. This is why interpolation terms $\tilde{\delta}_{j,r}(h)=\frac{1}{|\Gamma_{jr}|}\int_{\Gamma_{j,r}}h-\frac{h(\mathbf{x}_r)+h(\mathbf{x}_{r+1})}{2}$ are introduced. One gets after an algebraic manipulation

$$\begin{aligned}
D_4 &= \frac{1}{\varepsilon}\sum_j\sum_r|\Gamma_{j,r}|p_j^\varepsilon\Big(\tilde{\mathbf{n}}_{j,r},\tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\Big)+\frac{1}{\varepsilon}\sum_j\sum_r|\Gamma_{j,r}|\Big(\mathbf{u}_j^\varepsilon,\tilde{\mathbf{n}}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big)+\frac{\sigma}{\varepsilon^2}\sum_j\Big(\mathbf{u}_j^\varepsilon,\int_{\Omega_j}\mathbf{u}^\varepsilon\Big)\\
&+\frac{1}{\varepsilon}\sum_j\sum_r|\Gamma_{j,r}|p_j^\varepsilon\Big(\tilde{\mathbf{n}}_{j,r},\frac{\mathbf{u}^\varepsilon(\mathbf{x}_r)+\mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2}\Big)+\frac{1}{\varepsilon}\sum_j\sum_r|\Gamma_{j,r}|\Big(\mathbf{u}_j^\varepsilon,\tilde{\mathbf{n}}_{j,r}\frac{p^\varepsilon(\mathbf{x}_r)+p^\varepsilon(\mathbf{x}_{r+1})}{2}\Big)
\end{aligned}$$

By definition (56), $\mathbf{n}_{jr}l_{jr}=\frac{\tilde{n}_{j,r}|\Gamma_{j,r}|+\tilde{n}_{j,r-1}|\Gamma_{j,r-1}|}{2}$, so one can see that

$$\sum_j\sum_r|\Gamma_{j,r}|p_j^\varepsilon\Big(\tilde{\mathbf{n}}_{j,r},\frac{\mathbf{u}^\varepsilon(\mathbf{x}_r)+\mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2}\Big)=\sum_j\sum_r l_{jr}p_j^\varepsilon\big(\mathbf{n}_{jr},\mathbf{u}^\varepsilon(\mathbf{x}_r)\big).$$

It yields a slightly simpler expression

$$D_4 = \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\tilde{\mathbf{n}}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\right) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\right) + \frac{\sigma}{\varepsilon^2}\sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon\right)$$
$$+ \frac{1}{\varepsilon}\sum_j\sum_r l_{jr} p_j^\varepsilon\left(\mathbf{n}_{jr}, \mathbf{u}^\varepsilon(\mathbf{x}_r)\right) + \frac{1}{\varepsilon}\sum_j\sum_r l_{jr} p^\varepsilon(\mathbf{x}_r)\left(\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon\right)$$

One can now compute the sum $D_3 + D_4$

$$D_3 + D_4 = \frac{1}{\varepsilon}\sum_j\sum_r \left(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\right)\delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon}\sum_j\sum_r \left(l_{j,r}\mathbf{n}_{j,r}(p_{j,r}^\varepsilon - p_j^\varepsilon)\,,\ \delta_{j,r}(\mathbf{u}^\varepsilon)\right)$$
$$+ \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{jr}| p_j^\varepsilon \left(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\right) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{jr}| \left(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\right)$$
$$+ \frac{\sigma}{\varepsilon^2}\left(\sum_r\sum_j \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right).$$

One finally gets after rearrangement the final result (82) for $\mathscr{E}'(t) = D_1 + D_2 + D_3 + D_4$. □

The proof of the dissipative identity relies on a careful and technical evaluation of E_1, E_2 and E_3. Using the damping of the first term in (82), it is sufficient to obtained the desired result. We refer the reader to the appendix for all details.

3.6 Study of $\|\mathbf{DA}_h^\varepsilon - \mathbf{P}^0\|$

This main result in this section is the following.

Lemma 3.8. *There exists a constant $C_\downarrow$ independent of h and ε, with a growth in time less than $T^{\frac{3}{2}}$ such that one has the estimate*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0,T]\times\Omega))} \le C_\downarrow(h+\varepsilon)\|p_0\|_{H^4(\Omega)}. \tag{83}$$

This result, which is merely a consequence of (96) and (97) in proposition 3.12, will be obtained after studying in details the well-posedness, stability and consistency of the new diffusion asymptotic scheme rewritten after a convenient rescaling. The proof is provided just after the proof of the proposition. Additional technical results will be derived at the end of the section.

3.6.1 Rescaling of the equations

We rescale the semi-discrete diffuse asymptotic scheme $\mathbf{DA}_h^\varepsilon$ (63) wherein for convenience we made the following change of unknowns

$$\overline{\mathbf{u}}_r^\varepsilon = \frac{\mathbf{u}_r^\varepsilon}{\varepsilon} \text{ and } \overline{\mathbf{u}}_j^\varepsilon = \frac{\mathbf{u}_j^\varepsilon}{\varepsilon}. \tag{84}$$

In order to keep a simple notation we dropped the superscript ε and the bars. Thus the scheme (63) is now written as:

$$\begin{cases} |\ \Omega_j\ | \dfrac{d}{dt}p_j + \sum_r (l_{jr}\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_r - \mathbf{u}_j)\mathbf{n}_{jr} = 0 \\ \left(\varepsilon\sum_j \widehat{\alpha}_{jr} + \sigma B_r\right)\mathbf{u}_r = \sum_j l_{jr}p_j\mathbf{n}_{jr} + \varepsilon\sum_j \widehat{\alpha}_{jr}\mathbf{u}_j \end{cases} \tag{85}$$

Remark 3.9. *If wet set $\varepsilon = 0$ we naturally recover the limit diffusion scheme (62).*

This way of writing the system is much better to help the intuition, since it is can be naturally interpreted as the discretization of a diffusion equation.

3.6.2 Well-posedness

What we mean about well-posedness is the following: if we are able to write the last two relations of (85) as a non singular linear system with the $\mathbf{u}_r$'s and $\mathbf{u}_j$'s as unknowns, then we have a unique solution in terms of the p_j's. This notion is the relevant one for numerical discretization.

Let us denote $Y = (\{\mathbf{u}_j\}, \{\mathbf{u}_r\})$ the vector of unknowns. We can write the last two relations of (85) as $MY = b$ where M is a $(J+R)^2$ square matrix, J is the number of cells and R. One can observe that unless $\varepsilon = 0$, M is not a blockwise triangular matrix. One has

$$(MY, Y) = \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right)^2$$

Assume $(MY, Y) = 0$: in this case the geometrical assumption (70) implies that all the $\mathbf{u}_r$ are null and therefore it remains to study $\sum_j \sum_r l_{jr} \left(\mathbf{u}_j, \mathbf{n}_{jr}\right)^2 = 0$ that is $\sum_j (\mathbf{u}_j, C_j \mathbf{u}_j) = 0$ where $C_j = \sum_r l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$. Since the C_j are all invertible unless the mesh is degenerate, all the $\mathbf{u}_j$ are null: we have proved the invertibility of the matrix M and thus the scheme (85) exists and is uniquely defined.

3.6.3 Stability

We note $E(t) = \dfrac{1}{2} \sum_j |\Omega_j| p_j^2$. The initial data is $p_h(0) = (p_j(0))_{j \in Cells}$.

Proposition 3.10. *Under the geometrical assumption (70), the diffusion approximation scheme (85) is stable in the L^2 norm, in the sense that $E'(t) \leq 0$. One has*

$$||\mathbf{u}_r||_{L^2([0,T]\times\Omega)} \leq \frac{1}{\alpha} \|p_h(0)\|_{L^2(\Omega)} \tag{86}$$

and

$$\varepsilon \int_{[0,T]} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j - \mathbf{u}_r))^2 \leq \|p_h(0)\|_{L^2(\Omega)} . \tag{87}$$

Proof. One has

$$E^{'}(t) = \sum_j |\Omega_j| p_j \frac{d}{dt} p_j = -\sum_j p_j \sum_r (l_{jr} \mathbf{u}_r, \mathbf{n}_{jr}) = \sum_r \left(\mathbf{u}_r, \sum_j l_{jr} \mathbf{n}_{jr} p_j \right).$$

With the last equation of (85), one finds $E^{'}(t) = -\sum_r \left(\mathbf{u}_r, \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r \right) \mathbf{u}_r - \varepsilon \sum_j \widehat{\alpha}_{jr} \mathbf{u}_j \right)$. We expand the right hand side $E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_r \left(\mathbf{u}_r, \sum_j \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j \right)$. Permuting the sums in the second term of the right hand side , we show that

$$E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r (\mathbf{u}_r, \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j)) . \tag{88}$$

Using the definition of the $\mathbf{u}_j$, second line of (85), one has

$$\sum_j \left(\mathbf{u}_j, \sum_r \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j) \right) = 0. \tag{89}$$

Combining (89)$\times\varepsilon$ with (88) and using the definition of the matrices $\widehat{\alpha}_{jr}$ one has finally

$$E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right)^2 .$$

By the geometrical assumption (70) we have $E^{'}(t) \leq 0$, that is the L^2 stability. By integrating this equality on $[0,T]$ we obtain

$$E(T) + \int_0^T \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \int_0^T \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right)^2 = E(0)$$

Using again the geometrical assumption (70) for the terms $(B_r \mathbf{u}_r, \mathbf{u}_r)$ we have

$$E(T) + \alpha \int_0^T \sum_r |V_r| \, \|\mathbf{u}_r\|^2 + \int_0^T \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right)^2 \leq E(0)$$

which gives (86) and (87). $\square$

3.6.4 Consistency

For convenience we set

$$\overline{p}_j = p(\mathbf{x}_j, t) \quad \overline{\mathbf{u}}_j = -\frac{1}{\sigma} \nabla p(\mathbf{x}_j, t) \quad \overline{\mathbf{u}}_r = -\frac{1}{\sigma} \nabla p(\mathbf{x}_r, t) \tag{90}$$

where $p(x,t)$ is the solution of the diffusion equation. We define three consistency errors by inserting these quantities into the three equations of (85) which are also rescaled by a factor $\frac{1}{|\Omega_j|}$, $\frac{1}{h}$ and $\frac{1}{|V_r|}$. It yields

$$\begin{cases} a_j = \dfrac{d}{dt}\overline{p}_j + \dfrac{1}{\mid \Omega_j \mid} \displaystyle\sum_r (l_{jr} \overline{\mathbf{u}}_r, \mathbf{n}_{jr}), \\ \mathbf{b}_j = \dfrac{1}{h} \displaystyle\sum_r l_{jr} (\mathbf{n}_{jr}, \overline{\mathbf{u}}_r - \overline{\mathbf{u}}_j) \mathbf{n}_{jr}, \\ \mathbf{c}_r = \dfrac{1}{|V_r|} \left(\sigma B_r \overline{\mathbf{u}}_r - \displaystyle\sum_j l_{jr} \overline{p}_j \mathbf{n}_{jr} + \varepsilon \sum_j \widehat{\alpha}_{jr} (\overline{\mathbf{u}}_r - \overline{\mathbf{u}}_j) \right). \end{cases}$$

Proposition 3.11. *There exists a constant $C_c > 0$ independent on h and ε such that the following estimates hold*

$$\|a_h\|_{L^\infty([0,T]:L^2(\Omega))} \leq C_c h \|p_0\|_{H^4(\Omega)}, \tag{91}$$

$$\|\mathbf{b}_h\|_{L^\infty([0,T]:L^2(\Omega))} \leq C_c h \|p_0\|_{H^3(\Omega)}, \tag{92}$$

and

$$\|\mathbf{c}_h\|_{L^\infty([0,T]:L^2(\Omega))} \leq C_c (h + \varepsilon) \|p_0\|_{H^3(\Omega)}. \tag{93}$$

Proof. The proof uses the inequalities of proposition 3.3. For example one has

$$a_j = \frac{1}{\sigma} \left(\underbrace{\Delta p(\mathbf{x}_j, t) - \frac{\int_{\Omega_j} \Delta p(\mathbf{x}, t) dx}{|\Omega_j|}}_{=d_j^1} \right) + \frac{1}{\sigma |\Omega_j|} \left(\underbrace{\int_{\partial \Omega_j} \partial_n p d\tau - \sum_r l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t)\right)}_{=d_j^2} \right).$$

The first term is $|d_j^1| \leq C \, \|p\|_{H^4(\Omega_j)}$ by virtue of the first inequality of the proposition (3.3) with $\mathbf{x}_r$ changed into $\mathbf{x}_j$. The second term d_j^2 can be rearranged. Indeed by definition of $l_{jr} \mathbf{n}_{jr}$ one has

$$\sum_r l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t)\right) = \sum_k \int_{\partial \Omega_{jk}} \left(\frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\tau$$

where $n_j = \tilde{n}_{j,r}$ defined in the previous part and the nodes x_{jk}^+ and x_{jk}^- are the end of the edge $\partial\Omega_{jk} = \Omega_j \bigcap \Omega_k$, with the relation $\partial\Omega_j = \bigcup \partial\Omega_{jk}$. Therefore

$$d_j^2 = \sum_k \int_{\partial\Omega_{jk}} \left(\nabla p - \frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\tau.$$

The second inequality of the proposition 3.3 yields $|d_j^2| \le C_{\mathcal{A}} h^2 \|p\|_{H^4(\Omega_j)}$ Therefore one can write $a_j \le \widetilde{C}\|p_0\|_{H^4(\Omega_j)}$ where the constant is uniform with respect to j. It yields

$$\|a_h\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j| a_j^2} \le \sqrt{\sum_j |\Omega_j| C^2 \|p\|^2_{H^4(\Omega_j)}} \le Ch\|p\|_{H^4(\Omega)} \le Ch\|p_0\|_{H^4(\Omega)}. \tag{94}$$

Since it is true at any time t, it yields the first bound (91). The second inequality can be obtained with the same argument. Consider the decomposition

$$\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j) = \left(\nabla p(\mathbf{x}_r) - -\frac{\int_{\Omega_j} \nabla p(\mathbf{x}) dv}{|\Omega_j|} \right) - \left(\nabla p(\mathbf{x}_j) - -\frac{\int_{\Omega_j} \nabla p(\mathbf{x}) dv}{|\Omega_j|} \right).$$

Each parenthesis can be estimated with the first inequality of proposition 3.3. The rest of the proof of the second bound (92) is immediate since l_{jr} is neutralized by the $\frac{1}{h}$. The third bound is analyzed as follows. We write $\mathbf{c}_r = \mathbf{c}_r^a + \mathbf{c}_r^b$ with

$$\begin{aligned}
\mathbf{c}_r^a &= \frac{1}{|V_r|} \left(\left(\sigma \sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right) \left(-\frac{\nabla p(\mathbf{x}_r, t)}{\sigma} \right) - \sum_j l_{jr} \mathbf{n}_{jr} p(\mathbf{x}_j, t) \right) \\
&= \frac{1}{|V_r|} \sum_j \Big((\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) \Big) l_{jr} \mathbf{n}_{jr} \\
&= \frac{1}{|V_r|} \sum_j \Big((\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) + p(\mathbf{x}_r, t) \Big) l_{jr} \mathbf{n}_{jr}
\end{aligned}$$

and $\mathbf{c}_r^b = \frac{\varepsilon}{\sigma|V_r|} \left(\sum_j l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \left(\nabla p(\mathbf{x}_j, t) - \nabla p(\mathbf{x}_r, t) \right) \right)$. The first interpolation of proposition 3.3 can be used to evaluate the difference of point values in $\mathbf{c}_r^b$. It yields $|\mathbf{c}_r^b| \le C\frac{\varepsilon}{h}\|p\|_{H^3(\Omega_j)}$. Concerning $\mathbf{c}_r^a$ we notice that

$$(\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) + p(\mathbf{x}_r, t) = \left(\frac{1}{|\mathbf{x}_r - \mathbf{x}_j|} \int_{\mathbf{x}_j}^{\mathbf{x}_r} \nabla p(\mathbf{x}) d\tau - \nabla p(\mathbf{x}_r), \mathbf{x}_r - \mathbf{x}_j \right)$$

where the integral in along the chord between $\mathbf{x}_j$ and $\mathbf{x}_j$. The first term in the scalar product is the comparison between a mean value and a point value. So it can be estimated as in proposition 3.3. It yields similarly

$$\left| \left(\frac{1}{|\mathbf{x}_r - \mathbf{x}_j|} \int_{\mathbf{x}_j}^{\mathbf{x}_r} \nabla p(\mathbf{x}) d\tau - \nabla p(\mathbf{x}_r), \mathbf{x}_r - \mathbf{x}_j \right) \right| \le Ch\|p\|_{H^3(\Omega_j)}. \tag{95}$$

Thus $|\mathbf{c}_r^a| \le Ch\|p\|_{H^3(\Omega_j)}$. After summation of the $\mathbf{c}_r^a$s and $\mathbf{c}_r^b$s, one gets the last inequality of the claim. The constant C_c is the maximum of the three constants that show up in the three inequalities. □

3.6.5 Convergence

We study the numerical error between the solution of the diffusion asymptotic scheme written as (85) and the point values of the exact solution (90). Let us define three error variables

$$e_j = p_j - \overline{p}_j \ , \ \mathbf{f}_r = \mathbf{u}_r - \overline{\mathbf{u}}_r \text{ and } \mathbf{g}_j = \mathbf{u}_j - \overline{\mathbf{u}}_j$$

Proposition 3.12. *There exists constants $C_1 > 0$, $C_2 > 0$,$C_3 > 0$ and $C_4 > 0$ independent of h and ε, bounded for any time T and growing at most as $T^{\frac{3}{2}}$, such that*

$$\|e_h\|_{L^\infty([0,T]:L^2(\Omega))} \le C_1(h+\varepsilon)\|p_0\|_{H^4(\Omega)}, \tag{96}$$

$$\|\mathbf{f}_h\|_{L^2([0,T]\times\Omega)} \le C_2(h+\varepsilon)\|p_0\|_{H^4(\Omega)}, \tag{97}$$

and

$$\|\mathbf{g}_h\|_{L^2([0,T]\times\Omega)} \le C_3(h+\varepsilon)\sqrt{(1+\frac{h}{\varepsilon})}\|p_0\|_{H^4(\Omega)}. \tag{98}$$

Moreover

$$\sqrt{\varepsilon\int_0^T \sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2} \le C_4(h+\varepsilon)\|p_0\|_{H^4(\Omega)}. \tag{99}$$

Proof. By construction

$$\begin{cases} \mid \Omega_j \mid e_j' + \sum_r (l_{jr}\mathbf{f}_r,\mathbf{n}_{jr}) = - \mid \Omega_j \mid a_j \\ \sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{f}_r-\mathbf{f}_j)\mathbf{n}_{jr} = -h\mathbf{b}_j, \\ \left(\varepsilon\sum_j \widehat{\alpha}_{jr}+\sigma B_r\right)\mathbf{f}_r - \sum_j l_{jr}e_j\mathbf{n}_{jr} - \varepsilon\sum_j\widehat{\alpha}_{jr}\mathbf{f}_j = - \mid V_r \mid \mathbf{c}_r. \end{cases}$$

The errors are measured with $E(t) = \frac{1}{2}\|e_h\|^2_{L^2(\Omega)}$, $F(t) = \|\mathbf{f}_h\|^2_{L^2([0,t]\times\Omega)} = \int_0^t \sum_r |V_r|\,|\mathbf{f}_r|^2$ and $\|\mathbf{g}_h\|^2_{L^2([0,t]\times\Omega)} = \int_0^t\sum_j |\Omega_j|\;|\mathbf{f}_j|^2$. By proceeding as for the results of stability one has the identity

$$\begin{aligned} E'(t) &= \sum_j |\Omega_j| e_j e_j' = \sum_j e_j\left(-\left(\sum_r l_{jr}\left(\mathbf{n}_{jr},\mathbf{f}_r\right)\right) - |\Omega_j|\,a_j\right) \\ &= -\sum_r\sum_j (l_{jr}\mathbf{n}_{jr}e_j,\mathbf{f}_r) - \sum_j |\Omega_j|\,e_j a_j \\ &= -\sum_r\left(\mathbf{f}_r,\left(\varepsilon\sum_j\widehat{\alpha}_{jr}+\sigma B_r\right)\mathbf{f}_r - \varepsilon\sum_j\widehat{\alpha}_{jr}\mathbf{f}_j\right) - \sum_j|\Omega_j|\,e_j a_j - \sum_r |V_r|\left(\mathbf{c}_r,\mathbf{f}_r\right) \\ &= -\sum_r(\sigma B_r\mathbf{f}_r,\mathbf{f}_r) - \varepsilon\sum_r\left(\mathbf{f}_r,\sum_j\widehat{\alpha}_{jr}(\mathbf{f}_r-\mathbf{f}_j)\right) - \sum_j|\Omega_j|\,e_j a_j - \sum_r|V_r|\left(\mathbf{c}_r,\mathbf{f}_r\right) \\ &= -\sum_r(\sigma B_r\mathbf{f}_r,\mathbf{f}_r) - \varepsilon\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2 - \sum_j|\Omega_j|\,e_j a_j - \sum_r|V_r|\left(\mathbf{c}_r,\mathbf{f}_r\right) + \varepsilon\sum_j h(\mathbf{b}_j,\mathbf{f}_j). \end{aligned}$$

Using Young's inequality and assumptions (67) and (70), one gets

$$\begin{aligned} E'(t) \le &-\alpha\sigma\|\mathbf{f}_h\|^2_{L^2(\Omega)} - \varepsilon\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2 + \sqrt{2E(t)}||a_h||_{L^2(\Omega)} \\ &+\left(\frac{\mu}{2}||\mathbf{f}_h||^2_{L^2(\Omega)} + \frac{1}{2\mu}||\mathbf{c}_h||^2_{L^2(\Omega)}\right) + \frac{\varepsilon}{2hC_{\mathcal{M}}}\left(\eta||\mathbf{g}_h||^2_{L^2(\Omega)} + \frac{1}{\eta}||\mathbf{b}_h||^2_{L^2(\Omega)}\right) \end{aligned} \tag{100}$$

where $\mu,\eta > 0$ are two arbitrary coefficients that will be specified later. Now using (69) and (64), we have

$$|\Omega_j|\;|\mathbf{f}_j|^2 \le \frac{h}{\alpha}\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{f}_j)^2. \tag{101}$$

Therefore

$$|\Omega_j|\ |\mathbf{f}_j|^2 \leq \frac{h}{\alpha}\left(2\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{f}_j-\mathbf{f}_r)^2+2\sum_r l_{jr}|\mathbf{f}_r|^2\right),$$

which yields, using (68) and (65)

$$||\mathbf{g}_h||^2_{L^2(\Omega)} \leq \frac{2h}{\alpha}\sum_{jr} l_{jr}(\mathbf{n}_{jr},\mathbf{f}_j-\mathbf{f}_r)^2+\frac{2P}{\alpha C_{\mathcal{M}}}||\mathbf{f}_h||^2_{L^2(\Omega)}. \tag{102}$$

So from (100) we obtain

$$E'(t) \leq \sqrt{2E(t)}||a_h||_{L^2(\Omega)}+\frac{1}{2\mu}||\mathbf{c}_h||^2_{L^2(\Omega)}+\left(\frac{\mu}{2}+\frac{\varepsilon P\eta}{hC^2_{\mathcal{M}}\alpha}-\sigma\alpha\right)\|\mathbf{f}_h\|^2_{L^2(\Omega)}$$
$$+\left(\frac{\eta}{C_{\mathcal{M}}\alpha}-1\right)\varepsilon\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2+\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}||\mathbf{b}_h||^2, \qquad \forall\mu,\eta>0.$$

Let us choose the free coefficients μ and η so that

$$\frac{\mu}{2}+\frac{\varepsilon P\eta}{hC^2_{\mathcal{M}}\alpha}-\sigma\alpha \leq -\frac{\sigma\alpha}{4} \text{ and } \frac{\eta}{C_{\mathcal{M}}\alpha}-1\leq -\frac{1}{2}.$$

Let us choose first $\mu=\frac{\alpha\sigma}{2}$. The two inequalities reduce to

$$\frac{\varepsilon P\eta}{hC^2_{\mathcal{M}}\alpha}\leq\frac{\sigma\alpha}{2} \text{ adn } \frac{\eta}{C_{\mathcal{M}}\alpha}\leq\frac{1}{2}.$$

A natural solution is $\eta=\frac{C_{\mathcal{M}}\alpha}{2}\min\left(1,\frac{\alpha\sigma hC_{\mathcal{M}}}{\varepsilon P}\right)$. So

$$E'(t) \leq \sqrt{2E(t)}||a_h||_{L^2(\Omega)}-\frac{\alpha\sigma}{4}F'(t)-\frac{\varepsilon}{2}\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2$$
$$+\frac{1}{\alpha\sigma}||\mathbf{c}_h||^2_{L^2(\Omega)}+\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}||\mathbf{b}_h||^2_{L^2(\Omega)}.$$

By the consistency estimates (91-92-93), one has

$$\frac{1}{2}||a_h||^2_{L^2(\Omega)}+\frac{1}{\alpha\sigma}||\mathbf{c}_h||^2_{L^2(\Omega)}+\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}||\mathbf{b}_h||^2_{L^2(\Omega)}$$
$$\leq C_c^2\left(\frac{1}{2}h^2+\frac{1}{\alpha\sigma}(h+\varepsilon)^2+\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}h^2\right)\|p\|^2_{L^\infty([0,T]:H^4(\Omega))}$$
$$\leq C_c^2\left(\frac{1}{2}h^2+\frac{1}{\alpha\sigma}(h+\varepsilon)^2+\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}h^2\right)\|p_0\|^2_{H^4(\Omega)}.$$

The last term in the parenthesis is

$$\frac{\varepsilon}{2hC_{\mathcal{M}}\eta}h^2=\frac{1}{C^2_{\mathcal{M}}\alpha}\varepsilon h\max\left(1,\varepsilon P/(\alpha\sigma hC_{\mathcal{M}})\right)$$
$$\leq\frac{1}{C^2_{\mathcal{M}}\alpha}\varepsilon h\left(1+\varepsilon P/(\alpha\sigma hC_{\mathcal{M}})\right)=\frac{1}{C^2_{\mathcal{M}}\alpha}\varepsilon h+\frac{P}{C^3_{\mathcal{M}}\alpha^2\sigma}\varepsilon^2.$$

So there exists a constant C_e independent of h and ε such that

$$E'(t)\leq\sqrt{2E(t)}-\frac{\alpha\sigma}{4}F'(t)-\frac{\varepsilon}{2}\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2+C_e(h+\varepsilon)^2\|p_0\|^2_{H^4(\Omega)}. \tag{103}$$

Integrating (103), we find that for any for $t\leq T$

$$E(t)+\frac{\alpha\sigma}{4}F(t)+\frac{\varepsilon}{2}\int_0^t\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2\leq E(0)+\int_0^t\sqrt{2E(s)}||a_h||_{L^2(\Omega)}ds+tC_e(h+\varepsilon)^2\|p_0\|^2_{H^4(\Omega)}. \tag{104}$$

that is

$$\begin{aligned}
&E(t)+\frac{\alpha\sigma}{4}F(t)+\frac{\varepsilon}{2}\int_0^t\sum_j\sum_r l_{jr}\left(\mathbf{f}_r-\mathbf{f}_j,\mathbf{n}_{jr}\right)^2 \\
&\le E(0)+\int_0^t\sqrt{2E(s)}\sqrt{C_e}(h+\varepsilon)ds+TC_e(h+\varepsilon)^2\|p_0\|^2_{H^4(\Omega)}.
\end{aligned} \tag{105}$$

With another use of the Bihari's inequality, lemma (B.2), we obtain

$$\int_0^T E(t)\le\frac{1}{2}T\left(2\sqrt{E(0)+TC_e(h+\varepsilon)^2\|p_0\|^2_{H^4(\Omega)}}+T\sqrt{2C_e}(h+\varepsilon)\|p_0\|_{H^4(\Omega)}\right)^2 \tag{106}$$

By construction $E(0)\le C_{\mathcal{A}}^2h^2\|p_0\|^2_{H^2(\Omega)}$, which comes from inequality (71) which compares mean value and point value. $E(t)\le C_1^2(t)(h+\varepsilon)^2\|p_0\|^2_{H^4(\Omega)}$, where the constant C_1 is bounded for any T and growing as $T^{\frac{3}{2}}$. It gives (96), and one easily obtains (97) and (99) from (105) and the constants C_2 and C_3 are bounded for any time T and behave as a linear polynomial in T. Integrating (102) and using the estimates (97) and (99), one gets

$$\int_0^T ||\mathbf{g}||^2_{L^2(\Omega)}\le\frac{2}{\alpha}h\int_0^T\sum_{jr}l_{jr}(\mathbf{n}_{jr},\mathbf{f}_j-\mathbf{f}_r)^2+\frac{2P}{\alpha C_{\mathcal{M}}}||\mathbf{f}||^2_{L^2([0,T]\times\Omega)}\le C_4^2(h+\varepsilon)^2(1+\frac{h}{\varepsilon})\|p_0\|_{H^4(\Omega)},$$

where the constant C_4 is uniform in h and ε and is bounded for any T with, at most, and behave as a linear polynomial in T. The proof is finished. □

Proof of lemma 3.8. The norm of the estimate in the lemma 3.8 can be bounded from the sum of (96) and (98). However one must rescale back (98) since it corresponds to the scaled variable (84). This is why (98) must be multiplied by ε. It eliminates the $\varepsilon^{-\frac{1}{2}}$ divergence in (98). The constant $C_\downarrow\max(C_1,C_2)$ is bounded for any time T and behaves as $T^{\frac{3}{2}}$ for large T since it is the case for C_1. and ends the proof. □

3.6.6 Technical estimates

These technical estimates are needed in the next section. These results compare two different velocities at the initialization stage: on the one hand the velocity computed as the solution of the linear system made of the two last equations of (85), on the other hand the exact point wise velocity.

Proposition 3.13. *There exists a constant C independent of h and ε such that*

$$\left\|\left(\mathbf{u}_r+\frac{1}{\sigma}\nabla p(\mathbf{x}_r)\right)(t=0)\right\|_{L^2(\Omega)}\le C\sqrt{h\max(h,\varepsilon)}\|p_0\|_{H^3(\Omega)} \tag{107}$$

and

$$\left\|\left(\mathbf{u}_j+\frac{1}{\sigma}\nabla p(\mathbf{x}_j)\right)(t=0)\right\|_{L^2(\Omega)}\le C\sqrt{\frac{h}{\varepsilon}}\max(h,\varepsilon)\|p_0\|_{H^3(\Omega)}. \tag{108}$$

Proof. Let us write $\mathbf{q}_r=\mathbf{u}_r+\frac{1}{\sigma}\nabla p(\mathbf{x}_r)$ and $\mathbf{s}_j=\mathbf{u}_j+\frac{1}{\sigma}\nabla p(\mathbf{x}_j)$. These quantities are solution of the system

$$\left\{\begin{array}{llll}
\left(\varepsilon\sum_j l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr}+\sigma B_r\right)\mathbf{q}_r & -\varepsilon\sum_j l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr}\mathbf{s}_j & =\mathbf{d}_r^1+\mathbf{d}_r^2, & \forall r,\\
-\varepsilon\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{q}_r)\mathbf{n}_{jr} & +\varepsilon\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{s}_j)\mathbf{n}_{jr} & =\mathbf{d}_j, & \forall j,
\end{array}\right.$$

where the right hand sides are

$$\mathbf{d}_r^1=\sum_j l_{jr}p(\mathbf{x}_j)\mathbf{n}_{jr}+\sum_j l_{jr}(\mathbf{x}_r-\mathbf{x}_j,\nabla p(\mathbf{x}_r))\mathbf{n}_{jr},$$

$$\mathbf{d}_r^2 = \varepsilon \sum_j l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right) \mathbf{n}_{jr}$$

and

$$\mathbf{d}_j = -\varepsilon \sum_r l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right).$$

The right hand side $\mathbf{d}_r^1$ can be interpreted as a consistency error. Indeed it can be rewritten as

$$\mathbf{d}_r^1 = \sum_j \left[p(\mathbf{x}_j) - p(\mathbf{x}_r) + (\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r)]\right) l_{jr}\mathbf{n}_{jr},$$

one obtains from (95) the bound $|\mathbf{d}_r^1| \leq \sum_{j \text{ neighboring } r} \left[\widetilde{C}h\|p\|_{H^3(\Omega_j)}\right] h$. It yields after summation

$$\|\mathbf{d}^1\|_{L^2(\Omega)} \leq Ch^3\|p\|_{H^3(\Omega)}, \qquad C = \widetilde{C}P. \tag{109}$$

Taking the scalar product of the first line by $\mathbf{q}_r$ and of the second line by $\mathbf{s}_j$, one gets the identity

$$\sigma \sum_r (B_r\mathbf{q}_r, \mathbf{q}_r) + \varepsilon \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2$$

$$= \sum_r \left(\mathbf{d}_r^1, \mathbf{q}_r\right) + \varepsilon \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right) \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)$$

where $\mathbf{d}^1$ shows up explicitly. A Young's inequality yields

$$\sigma \sum_r (B_r\mathbf{q}_r, \mathbf{q}_r) + \frac{\varepsilon}{2} \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2 \leq \sum_r \left(\mathbf{d}_r^1, \mathbf{q}_r\right) + \frac{\varepsilon}{2} \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)^2. \tag{110}$$

The first term in the right hand side is

$$\sum_r \left(\mathbf{d}_r^1, \mathbf{q}_r\right) = \sum_r h^2 \left(\frac{1}{h^2}\mathbf{d}_r^1, \mathbf{q}_r\right) \leq C\frac{1}{h^2}\|\mathbf{d}^1\|_{L^2(\Omega)}\|\mathbf{q}\|_{L^2(\Omega)} \leq Ch\|p\|_{H^3(\Omega)}\|\mathbf{q}\|_{L^2(\Omega)}.$$

A similar reasoning as for (109), which one more time can be viewed as a consequence of the first technical inequality of proposition (3.3), is

$$\sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)^2 \leq Ch\|p\|_{H^3(\Omega)}^2.$$

So (110) implies (after redefinition of the constants)

$$\|\mathbf{q}\|_{L^2(\Omega)}^2 \leq C\left(\|\mathbf{q}\|_{L^2(\Omega)}\|p\|_{H^3(\Omega)} + \varepsilon\|p\|_{H^3(\Omega)}^2\right) h.$$

It means that $z = \frac{\|\mathbf{q}\|_{L^2(\Omega)}}{\|p\|_{H^3(\Omega)}}$ is below the maximal root of the polynomial $p(z) = z^2 - Chz - C\varepsilon h$, that is for some constant $K > 0$ $z \leq x^+ = \frac{Ch+\sqrt{C^2h^2+4C\varepsilon h}}{2} \leq K\sqrt{\max(h^2, h\varepsilon)}$. Noticing that $\|p\|_{H^3(\Omega)} \leq \|p_0\|_{H^3(\Omega)}$, It finishes the proof of the first inequality (107). Concerning the second inequality, we start from (101) to show that

$$\begin{aligned}
\|\mathbf{s}\|_{L^2(\Omega)}^2 = \sum_j |\Omega_j||\mathbf{s}_j|^2 &\leq \frac{h}{\alpha} \sum_j \sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{s}_j\right)^2 \\
&\leq 2\frac{h}{\alpha} \sum_j \sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r\right)^2 + 2\frac{h}{\alpha} \sum_j \sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2 \\
&\leq 2\frac{1}{C_{\mathcal{M}}\alpha} \sum_r |V_r| \left|\mathbf{q}_r\right|^2 + 2\frac{h}{\alpha} \sum_j \sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2 \\
&\leq 2\frac{1}{C_{\mathcal{M}}\alpha} \|\mathbf{q}\|_{L^2(\Omega)}^2 + 2\frac{h}{\alpha} \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2.
\end{aligned}$$

The first term is natural bounded bounded using (110). The crux of the estimate is the second term which is naturally bounded by (107)

$$\sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2 \leq \frac{2}{\varepsilon} \left(K\sqrt{\max(h^2, h\varepsilon)}h + C\varepsilon h\right) \|p_0\|^2_{H^3(\Omega)} \leq D(h+\varepsilon)\frac{h}{\varepsilon}\|p_0\|^2_{H^3(\Omega)}, \quad D > 0.$$

We obtain therefore

$$\|\mathbf{s}\|^2_{L^2(\Omega)} \leq C\left(\max(h^2, h\varepsilon) + h(h+\varepsilon)\frac{h}{\varepsilon}\right)\|p_0\|^2_{H^3(\Omega)}, \qquad C > 0.$$

The numbers h and ε can be considered less than 1. There are two cases: Either $h < \varepsilon$ so $\|\mathbf{s}\|^2_{L^2(\Omega)} \leq \widetilde{C}h\varepsilon\|p_0\|^2_{H^3(\Omega)}$ for another constant $\widetilde{C}$; or $\varepsilon \leq h$, so $\|\mathbf{s}\|^2_{L^2(\Omega)} \leq \widetilde{C}\frac{h^3}{\varepsilon}\|p_0\|^2_{H^3(\Omega)}$ for another constant $\widetilde{C}$. So we can writes $\|\mathbf{s}\|_{L^2(\Omega)} \leq C\sqrt{\frac{h}{\varepsilon}}\max(h,\varepsilon)$ for a certain constant $C > 0$ independent of h and ε. The proof of (108) is ended. □

Proposition 3.14. *There exists a constant C independent of h and ε such that*

$$\left\|\left(\frac{d}{dt}\mathbf{u}_r\right)\right\|_{L^2([0,T]\times\Omega)} \leq C\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\|p_0\|_{H^3(\Omega)}. \tag{111}$$

Proof. The proof is essentially a consequence of the previous proposition. Let us denote the time derivative of any f as $\tilde{f} = \partial_t f$. By linearity of the system (85), one has

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt}\tilde{p}_j + \displaystyle\sum_r (l_{jr}\tilde{\mathbf{u}}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr}(\mathbf{n}_{jr}, \tilde{\mathbf{u}}_r - \tilde{\mathbf{u}}_j)\mathbf{n}_{jr} = 0 \\ \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r\right)\tilde{\mathbf{u}}_r = \sum_j l_{jr}\tilde{p}_j\mathbf{n}_{jr} + \varepsilon\sum_j \widehat{\alpha}_{jr}\tilde{\mathbf{u}}_j \end{cases}$$

The L^2 stability property yields

$$\|\tilde{p}_h\|^2_{L^\infty([0,T];L^2(\Omega))} + \int_0^T \sum_r (B_r\tilde{\mathbf{u}}_r, \tilde{\mathbf{u}}_r)dt \leq \|\tilde{p}_h(0)\|^2_{L^2(\Omega)} \tag{112}$$

where this last quantity can be estimated with the first equation of (85): the square of the norm in (111) is also bounded by the same quantity. It remains to bound $\|\tilde{p}(0)\|_{L^2(\Omega))}$. Using again the notation $\mathbf{q}_r = \mathbf{u}_r + \frac{1}{\sigma}\nabla p(\mathbf{x}_r)$, we consider at time $t = 0$ the relation

$$\tilde{p}_j = \frac{d}{dt}p_j = -\frac{1}{|\Omega_j|}\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = \frac{1}{\sigma}\underbrace{\frac{1}{|\Omega_j|}\sum_r l_{jr}(\nabla p(\mathbf{x}_r), \mathbf{n}_{jr})}_{=v_j^1} - \underbrace{\frac{1}{|\Omega_j|}\sum_r l_{jr}(\mathbf{q}_r, \mathbf{n}_{jr})}_{=v_j^2}.$$

One has $v_j^1 = \frac{1}{|\Omega_j|}\sum_r l_{jr}(\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j), \mathbf{n}_{jr})$. Using techniques which have been used many times in this paper, one has $|v_j^1| \leq C\frac{1}{h}\|p\|_{H^3(\Omega_j)}$, which turns into

$$\|v^1\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j|(v_j^1)^2} \leq C\|p\|_{H^3(\Omega)} \leq C\|p_0\|_{H^3(\Omega)}, \qquad C > 0.$$

The other term is naturally bounded by the norm of $\mathbf{q}$, that is $\|v^2\|_{L^2(\Omega)} \leq \frac{P}{C_{\mathcal{M}}h}\|\mathbf{q}\|_{L^2(\Omega)}$, P the maximal number of vertices per cell. Going back to (107), one obtains

$$\|v^2\|_{L^2(\Omega)} \leq C\frac{1}{h}\sqrt{h\max(h,\varepsilon)}\|\mathbf{p}_0\|_{H^3(\Omega)} \leq C\sqrt{\max(1, \varepsilon/h)}\|\mathbf{p}_0\|_{H^3(\Omega)}. \tag{113}$$

The sum $\|v^1\|_{L^2(\Omega)} + \|v^2\|_{L^2(\Omega)}$ yields the bound for $\tilde{p}_h(0)$ that was looked for. The estimate is dominated by the worst term which is the right hand side of (113). Plugging in (112), the proof is finished. □

3.7 Study of $\|\mathbf{P}_h^\varepsilon - \mathbf{DA}_h^\varepsilon\|$

In this section we estimate the difference between the hyperbolic scheme $\mathbf{P}_h^\varepsilon$ and the diffusion asymptotic scheme $\mathbf{DA}_h^\varepsilon$. Since the discrete of the discrete equations are very similar, this proof is simple. This is where we get the clear benefit of the introduction of the new diffusion asymptotic scheme.

Lemma 3.15. *There exists a constant $C^\rightarrow$ independant of h and ε, with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\Omega))} \le C^\rightarrow \left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right) \|p_0\|_{H^3(\Omega)}. \tag{114}$$

Proof. We introduce $\mathbf{R}_j = \frac{d}{dt}\mathbf{u}_j$ such that the solution $\mathbf{V}_h$ of the diffusion scheme (63) satisfies

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} p_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr}\mathbf{n}_{jr}, \mathbf{u}_r) = 0, \\ \mid \Omega_j \mid \dfrac{d}{dt} \mathbf{u}_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} p_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \dfrac{\sigma}{\varepsilon} B_r\right) \mathbf{u}_r - \displaystyle\sum_j l_{jr} p_j \mathbf{n}_{jr} - \sum_j \widehat{\alpha}_{jr} \mathbf{u}_j = 0. \end{cases} \tag{115}$$

By definition $\|\mathbf{R}\|_{L^2(\Omega)} = \|\frac{d}{dt}\mathbf{u}_j\|_{L^2(\Omega)}$. Using the second line of (63), one has $\mathbf{u}_j = A_j^{-1}\sum_r \widehat{\alpha}_{jr}\mathbf{u}_r$ and thus $\|\frac{d}{dt}\mathbf{u}_j\|_{L^2(\Omega)} \le C\|\frac{d}{dt}\mathbf{u}_r\|_{L^2(\Omega)}$. Using (111) (and taking care that rescaling (84) by a factor ε was systematically used in the previous section), one gets for a smooth initial data

$$\|\mathbf{R}\|_{L^2([0,T]\times\Omega)} \le C\varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) \|p_0\|_{H^3(\Omega)}.$$

We denote by $e_j = p_j - p_j^\varepsilon$, $\mathbf{f}_j = \mathbf{u}_j - \mathbf{u}_j^\varepsilon$ and $\mathbf{f}_r = \mathbf{u}_r - \mathbf{u}_r^\varepsilon$. One finds, making the difference between the schemes (115) and (57):

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} e_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr}\mathbf{n}_{jr}, \mathbf{f}_r) = 0, \\ \mid \Omega_j \mid \dfrac{d}{dt} \mathbf{f}_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} e_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{f}_j - \mathbf{f}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \dfrac{\sigma}{\varepsilon} B_r\right) \mathbf{f}_r - \displaystyle\sum_j l_{jr} e_j \mathbf{n}_{jr} - \sum_j \widehat{\alpha}_{jr} \mathbf{f}_j = 0. \end{cases}$$

We are going to write an inequality satisfied by $E(t) = \|e(t)\|_{L^2(\Omega)}^2 + \|\mathbf{f}(t)\|_{L^2(\Omega)}^2$, knowing that $e(0) = 0$. Using the same kind of proof than for the L^2 stability of the JL-(b) scheme (proposition 3.6), one can show that

$$\frac{1}{2}\frac{d}{dt}E(t) \le \sum_j |\Omega_j|(\mathbf{R}_j, \mathbf{f}_j) \le \|\mathbf{f}\|_{L^2(\Omega)}\|\mathbf{R}\|_{L^2(\Omega)} \le \sqrt{E(t)}\|\mathbf{R}\|_{L^2(\Omega)}.$$

By integration, one has for $t \le T$

$$\sqrt{E(t)} \le \sqrt{E(0)} + \sqrt{T}\|\mathbf{R}\|_{L^2([0,T]\times\Omega)} = \|\mathbf{f}(0)\|_{L^2(\Omega)} + \sqrt{T}\|\mathbf{R}\|_{L^2([0,T]\times\Omega)}.$$

One has $\|\mathbf{f}(0)\|_{L^2(\Omega)} \le C\sqrt{h\varepsilon}\max(h,\varepsilon)\|p_0\|_{H^3(\Omega)}$ by virtue of (108) (taking care that there is a rescaling (84) by ε). We simplify a little $\|\mathbf{f}(0)\|_{L^2(\Omega)} \le C\left(h^2 + \varepsilon^2\right)\|p_0\|_{H^3(\Omega)}$, so

$$\sqrt{E(t)} \le C\left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\sqrt{T}\right) \|p_0\|_{H^3(\Omega)}, \qquad C > 0.$$

Since $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\Omega))} = \sqrt{\int_0^T E(t)dt}$, the proof is ended with

$$C^{\rightarrow} = CT\left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right)\|p_0\|_{H^3(\Omega)}$$

. $\square$

3.8 Space estimate for uniform AP property in 2D

We have the following result of uniform convergence for a mesh satisfying the geometrical assumptions (3.2).

Theorem 3.16 (Space estimate)**.** *There exists a constant $C>0$ independent of h and ε, increasing at most as $T^{\frac{3}{2}}$, such that the following estimate holds*

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \le Ch^{\frac{1}{4}}\|p_0\|_{H^4(\Omega)}.$$

Proof. The proof is a slight adaptation of our initial proposition 1.3, where we use the norm $\|\cdot\| = \|\cdot\|_{L^2([0,T]\times\Omega))}$. From the triangular inequality applied to the AP diagram, one has

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| \le \min\left(\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{\text{naive}}, \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\| + \|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\| + \|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\|\right).$$

All these norms are estimated with (75), (114), (83) and (74). Therefore one can write

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| \le C\min\left(\sqrt{\frac{h}{\varepsilon}}\ ,\ \left(h^2 + \varepsilon\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right) + (h+\varepsilon) + \varepsilon\right)\|p_0\|_{H^4(\Omega)}, \quad C>0,$$

where

$$C = \max\left[\frac{{}_{\downarrow}C}{\sqrt{C_{\mathcal{M}}}}, \frac{C^{\rightarrow}}{C_{\mathcal{M}}}, \frac{C_{\downarrow}}{\sqrt{C_{\mathcal{M}}}}, C_{\leftarrow}\right]$$

and behaves as $T^{\frac{3}{2}}$ for large T.

The parenthesis is

$$\mathcal{Z} = \min\left(\sqrt{\frac{h}{\varepsilon}}\ ,\ \left(h^2 + \varepsilon\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right) + (h+\varepsilon) + \varepsilon\right)$$

$$\le \min\left(\sqrt{\frac{h}{\varepsilon}}\ ,\ \varepsilon\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + 2h + 2\varepsilon\right) \le \min\left(\sqrt{\frac{h}{\varepsilon}}\ ,\ 3\varepsilon\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + 2h\right).$$

As in proposition 1.3, a threshold value is obtained by equating the more singular terms, that is $\sqrt{\frac{h}{\varepsilon_{\text{thresh}}}} = \varepsilon_{\text{thresh}}\sqrt{\frac{\varepsilon_{\text{thresh}}}{h}}$, with solution $\varepsilon_{\text{thresh}} = \sqrt{h}$. Two case occur. The first case is $\varepsilon \ge \varepsilon_{\text{thresh}}$. Then the first term in $\mathcal{Z}$ shows that $\mathcal{Z} \le \sqrt{\frac{h}{\varepsilon_{\text{thresh}}}} = h^{\frac{1}{4}}$. The second case is $\varepsilon \le \varepsilon_{\text{thresh}}$. Then the second term in $\mathcal{Z}$ shows that $\mathcal{Z} \le 3\varepsilon_{\text{thresh}}\sqrt{\frac{\varepsilon_{\text{thresh}}}{h}} + 2h = 3h^{\frac{1}{4}} + 2h \le 5h^{\frac{1}{4}}$. In both case $\mathcal{Z} \le Ch^{\frac{1}{4}}$. The proof is ended. $\square$

4 Implicit discretization and proof of theorem 1.1

We explain hereafter how to compare the implicit scheme and the semi-discrete scheme, in a way that produces immediately abstract error bounds. This technique comes from [12] where applications to the numerical analysis of explicit schemes was the main goal. In what follows we concentrate on implicit Euler discretization for two reasons. First reason is that the theory is a little simpler to explain than for the explicit scheme, for which the interested reader may nevertheless refer to the cited work. The very simple proof that is provided is new. Second reason is that implicit discretization is somehow necessary to take into the account the intrinsic stiffness of the problem. In particular the numerical tests have been performed with the implicit method. With the explicit method the CFL condition is so restrictive that it makes impossible the convergence study. The proof is a consequence of the abstract estimate (120) with the technical estimate (126) for the initial data.

4.1 An abstract estimate

The idea is to compare the solution $U_h(t)$ of a semi-discrete scheme

$$U_h(t) = A_h U_h(t), \quad U_h(0) = U_h^{\rm ini} \tag{116}$$

with the solution of the corresponding implicit Euler scheme with time step Δt

$$\frac{U_h^{n+1} - U_h^n}{\Delta t} = A_h U_h^{n+1}, \quad U_h^0 = U_h^{\rm ini} \tag{117}$$

The operator depends on an abstract parameter h: with symbolic notation, this abstract parameter is $h \leftarrow (h, \varepsilon)$ in the case of our problem $\mathbf{P}_h^\varepsilon$. The question is to bound the difference of these two uniformly with respect to Δt and uniformly with respect to the abstract parameter h.

We assume a natural L^2 norm denoted as $\|\cdot\|$ with the associated scalar product. For simplicity we also assume that A_h is dissipative in the sense that

$$(U_h, A_h U_h) \leq 0 \quad \text{ for all } U_h \text{ in the appropriate discrete space.}$$

Taking the scalar product of (117) with U_h^{n+1}, one deduces that $\|U_h^{n+1}\| \leq \|U_h^n\|$ for all U_h^n. Assuming the discrete space in finite (this is always true for discrete methods in a compact domain), one gets the unconditional stability estimate

$$\|(I_h - \Delta t A_h)^{-1}\| \leq 1 \qquad \forall \Delta t > 0 \tag{118}$$

where I_h is the discrete identity operator and the norm is the induced one for operators. Note that (118) ultimately shows that the matrix $I_h - \Delta t A_h$ is non singular. So the matrix of the problem can be assemble and invert on a computer.

Let us define for convenience $V_h^n = U_h(n\Delta t)$ so that the semi-discrete scheme can be rewritten as

$$\frac{V_h^{n+1} - V_h^n}{\Delta t} - A_h V_h^{n+1} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} U_h'(s) ds - A_h U_h((n+1)\Delta t)$$

$$= \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} A_h U_h(s) ds - A_h U_h((n+1)\Delta t) = \Delta t A_h s_h^{n+1}$$

where the residual is $s_h^{n+1} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{U_h(s) - U_h((n+1)\Delta t)}{\Delta t} ds$. We notice that

$$\|s_h^n\| \leq \sup_{0 \leq s \leq T} \|U_h'(s)\| \leq \|A_h U_h^{\rm ini}\|, \quad n\Delta t \leq T. \tag{119}$$

Therefore this special residual is uniformly bounded provided $\|A_h U_h^{\rm ini}\|$ is uniformly bounded. This is actually true: it comes from the fact that $W_h(t) = U_h'(t)$ is solution of $W_h'(t) = A_h W_h(t)$ and $W_h(0) = A_h U_h^{\rm ini}$. So the strong L^2 stability of the semi-discrete scheme due to (121) yields the bound (119).

Proposition 4.1 (Time estimate)**.** *Let $T > 0$ be a final time. Then there exists a constant C independent of h, ε and Δt, proportional to $\sqrt{T}$, such that*

$$\|U_h^n - U_h(n\Delta t)\| \leq C\sqrt{\Delta t}\|A_h U_h^{\rm ini}\|, \quad n\Delta t \leq T. \tag{120}$$

Proof. The initial data is the same $V_h^0 = U_h^{\rm ini}$. Let us define the error $E_h^n = V_h^n - U_h^n$ which is solution of

$$\frac{E_h^{n+1} - E_h^n}{\Delta t} = A_h E_h^{n+1} + \Delta t A_h s_h^{n+1}, \quad E_h^0 = 0. \tag{121}$$

It yields $(I_h - \Delta t A_h)E_h^{n+1} = E_h^n + \Delta t^2 A_h s_h^{n+1}$, that is $E_h^{n+1} = (I_h - \Delta t A_h)^{-1} E_h^n + \Delta t (I_h - \Delta t A_h)^{-1} \Delta t A_h s_h^{n+1}$. We obtain the representation formula (discrete Duhamel's formula)

$$E_h^n = \Delta t \sum_{p=0}^{n-1} \left[(I_h - \Delta t A_h)^{-1} \right]^{n-1-p} (I_h - \Delta t A_h)^{-1} \Delta t A_h s_h^{p+1}. \tag{122}$$

Let us define the operator $T_h = (I_h - \Delta t A_h)^{-1}$ which is bounded $\|T_h\| \leq 1$. One has the formula $T_h - I_h = (I_h - \Delta t A_h)^{-1} \Delta t A_h$ and the formula $\left(I_h - \frac{\Delta t}{2} A_h\right)^{-1} \frac{I_h + T_h}{2} = (I_h - \Delta t A_h)^{-1}$. Plugging in the discrete Duhamel's formula, one obtains another representation

$$E_h^n = \Delta t \sum_{p=0}^{n-1} \left[(I_h - \frac{\Delta t}{2} A_h)^{-(n-1-p)} \right] \left[\left(\frac{I_h + T_h}{2} \right)^{n-1-p} (T_h - I_h) \right] s_h^{p+1}. \tag{123}$$

The first operator in brackets is bounded by 1 due to the stability (118). On the other hand it is an easy exercise in number theory to show that for $q \geq 0$

$$\left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) = \frac{1}{2^q} \sum_r \left(\left(\begin{array}{c} q \\ r-1 \end{array} \right) - \left(\begin{array}{c} q \\ r \end{array} \right) \right) T_h^r$$

where the binomial coefficients are $\left(\begin{array}{c} q \\ r \end{array} \right) = \frac{q!}{r!(q-r)!}$ for $0 \leq r \leq q$, otherwise zero. Therefore

$$\left\| \left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) \right\| \leq \frac{1}{2^q} \sum_r \left| \left(\begin{array}{c} q \\ r-1 \end{array} \right) - \left(\begin{array}{c} q \\ r \end{array} \right) \right| \leq \frac{1}{2^q} 2 \left(\begin{array}{c} q \\ r_* \end{array} \right)$$

where the last inequality is from a telescoping reasoning and r_* is one of the closest entire number to $q/2$, that is $|\frac{q}{2} - r_*| \leq 1$. But there exists a universal constant, denoted K, such that $\frac{1}{2q-1} \left(\begin{array}{c} q \\ r_* \end{array} \right) \leq \frac{K}{\sqrt{q+1}}$. Therefore $\left\| \left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) \right\| \leq K/\sqrt{q+1}$. Using this universal estimate in (123) and the estimate on s_h^n, we obtain $\|E_h^n\| \leq \Delta t \sum_{p=0}^{n-1} \frac{K}{\sqrt{n-1-p+1}} \|A_h U_h^{\rm ini}\| = \Delta t \sum_{p=1}^n \frac{K}{\sqrt{p}} \|A_h U_h^{\rm ini}\|$. A basic bound shows that $\sum_{p=1}^n \frac{1}{\sqrt{p}} \leq \widetilde{K} \sqrt{n}$. Therefore

$$\|E_h^n\| \leq \Delta t K \widetilde{K} \sqrt{n} \|A_h U_h^{\rm ini}\| \leq (K \widetilde{K} \sqrt{T}) \sqrt{\Delta t} \|A_h U_h^{\rm ini}\|, \quad n\Delta t \leq T.$$

The proof is ended. □

To finish the proof of the theorem 1.1, it is now necessary and sufficient to show that $\|\frac{d}{dt} U_h(0)\| = \|A_h U_h^{\rm ini}\|$ is bounded independently of h for the initial data of $\mathbf{P}_h^\varepsilon$. This is the purpose of the next section.

4.2 Technical estimates

To prove the uniform on the initial data, we will use in a slightly different manner the estimates for the initial data that have been obtained for the diffusion approximation scheme $\mathbf{DA}_h^\varepsilon$. However we will need an additional assumption of the mesh

$$(A_r \mathbf{u}, \mathbf{u}) \geq \alpha h(\mathbf{u}, \mathbf{u}), \quad A_r = \sum_j \widehat{\alpha}_{jr}. \tag{124}$$

This assumption is not restrictive so we do not comment on it. The following technical estimates show two things. First in explains in what sense the corner velocity is a good approximation of the gradient at the corner at initial stage. Second it provides in (126) a control of the time derivative at time $t = 0$ uniformly with respect to h and ε, it immediately shows the boundedness of the abstract quantity $A_h U_h^{\mathrm{ini}}$ in (119). So it is possible to apply the above proposition and the main theorem is proved. We now turn to the proof the technical estimates.

Proposition 4.2. *There exists a constant C independent of h and ε such that the initial data of $\mathbf{P}_h^\varepsilon$ satisfies*

$$\left\| \mathbf{u}_r^\varepsilon(0) + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right\|_{L^2(\Omega)} \leq Ch\varepsilon \|p_0\|_{H^3(\Omega)}. \tag{125}$$

Proof. The corner problem (59) that defines $\mathbf{u}_r = \mathbf{u}_r^\varepsilon(0)$ at initial time is rewritten as

$$\left(A_r + \frac{\sigma}{\varepsilon} B_r\right) \mathbf{u}_r = \sum_j l_{jr} p_0(\mathbf{x}_j) \mathbf{n}_{jr} - \frac{\varepsilon}{\sigma} \sum_j \widehat{\alpha}_{jr} \nabla p_0(\mathbf{x}_j).$$

Let us defined $\mathbf{d}_r^1 = \sum_j l_{jr} \left(p_0(\mathbf{x}_j) - p_0(\mathbf{x}_r) - (\mathbf{x}_j - \mathbf{x}_r, \nabla p_0(\mathbf{x}_r))\right) \mathbf{n}_{jr}$, already defined and bounded in (109). So elimination of $p(\mathbf{x}_j)$ and simplification with $\sum_j l_{jr} \mathbf{n}_{jr} p(\mathbf{x}_r) = 0$ yield

$$\left(A_r + \frac{\sigma}{\varepsilon} B_r\right) \mathbf{u}_r = \sum_j l_{jr} \left(\mathbf{x}_r - \mathbf{x}_j, \nabla p_0(\mathbf{x}_r)\right) \mathbf{n}_{jr} + \mathbf{d}_r^1$$

$$- \frac{\varepsilon}{\sigma} \sum_j \widehat{\alpha}_{jr} \nabla p_0(\mathbf{x}_r) + \frac{\varepsilon}{\sigma} \sum_j \widehat{\alpha}_{jr} \left(\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)\right).$$

that is with the definition of the matrices

$$\left(A_r + \frac{\sigma}{\varepsilon} B_r\right) \left(\mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r)\right) = \mathbf{d}_r^1 + \frac{\varepsilon}{\sigma} \sum_j \widehat{\alpha}_{jr} \left(\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)\right).$$

The coercivity (124) of the matrices A_r and B_r yields

$$\alpha \left(h + \frac{\sigma h^2}{\varepsilon}\right) \left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq |\mathbf{d}_r^1| + \frac{\varepsilon}{\sigma} \sum_j \widehat{\alpha}_{jr} \left| \nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j) \right|.$$

With estimate of $\mathbf{d}_r^1$ (109), estimate of the difference $\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)$, it yields

$$\alpha \left(h + \frac{\sigma h^2}{\varepsilon}\right) \left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq C(h^2 + \frac{\varepsilon}{\sigma} h) \|p\|_{\Omega_j},$$

with a constant uniform with respect to h, ε and the index of the cell j. That is

$$\left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq \frac{C}{\alpha} \varepsilon \|p\|_{\Omega_j}.$$

After squaring and summation with respect to j, it yields the result. ∎

Proposition 4.3. *There exists a constant $C > 0$ which do not depend on h and ε such that the initial data of $\mathbf{P}_h^\varepsilon$ satisfies*

$$\left\| \frac{d}{dt} \mathbf{V}_h^\varepsilon \right\|_{L^2(\Omega))} \leq C \|p_0\|_{H^3(\Omega)}. \tag{126}$$

Proof. The $\mathbf{P}_h^\varepsilon$ scheme (57) or (61) can be rewritten

$$\mathbf{P}_h^\varepsilon : \quad \left\{ \begin{array}{ll} \dfrac{d}{dt} p_j^\varepsilon & = -\dfrac{1}{\varepsilon |\Omega_j|} \displaystyle\sum_r l_{jr} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon, \mathbf{n}_{jr}) \\ \dfrac{d}{dt} \mathbf{u}_j^\varepsilon & = -\frac{1}{\varepsilon |\Omega_j|} \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon) \mathbf{n}_{jr}. \end{array} \right. \tag{127}$$

At time $t=0$ one has $\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon = \left(\mathbf{u}_r^\varepsilon + \frac{\varepsilon}{\sigma}\nabla p_0(\mathbf{x}_r)\right) + \frac{\varepsilon}{\sigma}\left(\nabla p_0(\mathbf{x}_j) - \nabla p_0(\mathbf{x}_r)\right)$: the first term can be estimated by (125) and the second one as usual. Therefore there exists constants such that

$$\left\| \frac{1}{\varepsilon|\Omega_j|}\sum_r l_{jr}(\mathbf{u}_r^\varepsilon + \frac{\varepsilon}{\sigma}\nabla p_0(\mathbf{x}_r), \mathbf{n}_{jr}) \right\|_{L^2(\Omega)} \leq C\frac{h}{C_{\mathcal{M}}h^2}Ph\varepsilon\|p_0\|_{H^3(\Omega)} \leq \widehat{C}\|p_0\|_{H^3(\Omega)}.$$

In a similar way

$$\frac{1}{\sigma|\Omega_j|}\left|\sum_r l_{jr}(\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j), \mathbf{n}_{jr})\right| \leq \frac{h}{\sigma C_{\mathcal{M}}h^2}\widetilde{C}\|p_0\|_{H^3(\Omega_j)} \leq \frac{\overline{C}}{h}\|p_0\|_{H^3(\Omega_j)}.$$

Therefore

$$\left\| \frac{1}{\sigma|\Omega_j|}\sum_r l_{jr}(\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j), \mathbf{n}_{jr}) \right\|_{L^2(\Omega)} \leq \overline{C}\|p_0\|_{H^3(\Omega)}.$$

It shows that $\left\|\frac{d}{dt}p^\varepsilon(0)\right\|_{L^2(\Omega)} \leq C\|p_0\|_{H^3(\Omega_j)}$. Considering (127) , a similar result for $\frac{d}{dt}\mathbf{u}^\varepsilon(0)$. It shows $\left\|\frac{d}{dt}\mathbf{V}_h^\varepsilon(0)\right\|_{L^2(\Omega))} \leq C\|p_0\|_{H^3(\Omega)}$. The proof is ended. $\square$

5 Numerical illustration

To illustrate the theory and have a more quantitative version of the error estimates studied in this work, we consider the academic square $\Omega = [0,1]^2$ and discretize the hyperbolic heat equation of a mesh made with random quads. A random quad mesh is made of quads where the vertices are moved randomly around their initial position, by a factor between 10% and 30%. We use the fully implicit time discretization version of the 2D scheme detailed in this work. The solution of the linear systems is computed via an iterative GMRES algorithm, which converges smoothly in our numerical experiments. The reference analytical solution used in our tests is designed by separation of variables. A solution of (1) is

$$p = f + \frac{\varepsilon^2}{\sigma}\partial_t f \text{ and } \mathbf{u} = -\frac{\varepsilon}{\sigma}\nabla f,$$

with f solution of

$$\partial_t f + \frac{\varepsilon^2}{\sigma}\partial_t^2 f - \frac{1}{\sigma}\Delta f = 0. \tag{128}$$

We propose to construct a solution for a subset of small ε to validate the uniform convergence. Firstly we consider that the solution is a periodic solution on the square $[0,2]\times\left[0,\frac{2}{L}\right]$. For this we use the separation of the variables. We consider the following function

$$f(t,x,y) = \alpha(t)\cos(L\pi x))\cos(L\pi y).$$

and we propose to find the function $\alpha(t)$ such that $f(t,x,y) = \alpha(t)\cos(L\pi x)\cos(L\pi y)$ is a periodic solution of (128). The function α is determined as the solution of

$$\alpha'(t) + \frac{\varepsilon^2}{\sigma}\alpha''(t) + \frac{2L^2\pi^2}{\sigma}\alpha(t) = 0$$

with $\alpha'(0) = 0$ and $\alpha(0) = 1$. For small ε, which is the case we are interested in, the solution is computed as follows. First determine

$$\lambda_1 = -\frac{\sigma\left(\sqrt{1-\frac{\varepsilon^2}{\sigma^2}8L^2\pi^2}+1\right)}{2\varepsilon^2} \text{ and } \lambda_2 = \frac{\sigma\left(\sqrt{1-\frac{\varepsilon^2}{\sigma^2}8L^2\pi^2}-1\right)}{2\varepsilon^2}.$$

Then

$$\alpha(t) = \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2 t}$$

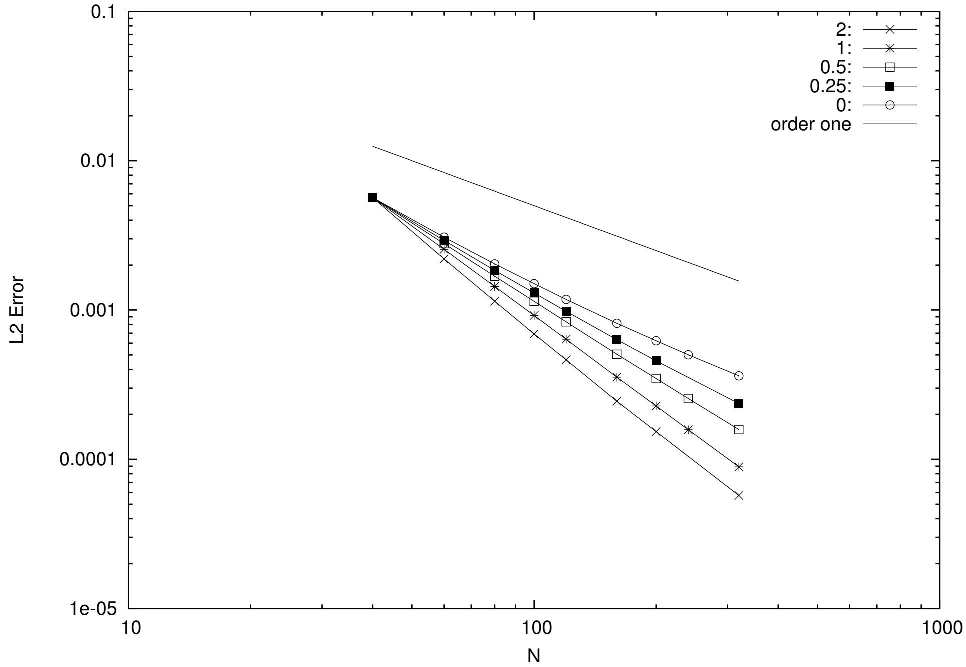

Figure 6: The error is plotted in log scale versus the number of cells per direction for the test problem described in section 5. Each curve corresponds to a value of $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$, plus a reference line for order one. One sees that the order of convergence is an increasing function of τ.

from which $p(t)$ and $\mathbf{u}(t)$ are easily recovered.

We decide that an exact relation is enforced between ε and $h = \frac{1}{N}$, so that the error can be expressed as a function of h solely. The relation between ε and h writes $\varepsilon = 0.01(40h)^\tau$ for $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$. The error between the exact solution and the numerical solution is computed numerically in function of $h = \frac{1}{N}$, for different values of τ, and the results of some of these numerical experiments is displayed in figure 6. The results correspond to the time $T = 0.02$ using the time step $\Delta t = 0.2h^2$.

As predicted by the theory, the scheme is uniformly AP and the error behavior is a continuous function of γ between the hyperbolic and parabolic limits. However the results are much better, in the sense the order is greater than the theoretical prediction since the order is approximatively 1 for $\gamma = 0$ (hyperbolic limit) and 2 for $\gamma = 2$ (parabolic regime). We can find a closed result on the second order convergence for the parabolic regime in the paper [1] (1D linear problem). The reason is probably that the theory is based on worst case estimates, as it is often the case for the numerical analysis of finite volume schemes [14].

6 Conclusion

The proof that was given of the uniform AP property is quite technical. It relies on specific hyperbolic and parabolic estimates for linear nodal finite volume schemes on general meshes. We observe that the multidimensional case yields an additional contribution in the error that ultimately slightly degrades the convergence rate. It is an open problem to determine if these inequalities are optimal. The numerical results indicate that it is probably not the case.

A Detailed proof of the naive estimate (75)

Our aim is to now examine each term in the right hand side of the dissipative identity (82). Its first term is already non positive.

Proposition A.1. *Let $\gamma > 0$ be a number which precise value will be determined further. There exists a constant $C_1(\gamma)$ which depends on γ such that one has the bound for the second term of the dissipative identity (82)*

$$\int_0^T E_1(t)dt \le \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\boldsymbol{n}_{j,r}, \boldsymbol{u}_j^\varepsilon - \boldsymbol{u}_r^\varepsilon)^2 + C_1(\gamma)\frac{h}{\varepsilon\sqrt{C_{\mathcal{M}}}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}. \tag{129}$$

Proof. We use a Young's inequality $ab \le \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2$,with some positive constantγ which will be defined later, for the second term and the definition of the fluxes (58) for the third term: we get

$$E_1 \le \frac{\gamma}{2\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon}\sum_j\sum_r \delta_{j,r}(p^\varepsilon)^2$$
$$+ \sum_j\sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)(\mathbf{n}_{jr}, \delta_{j,r}(\mathbf{u}^\varepsilon)) - \frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right)$$

Another use of Young's inequality with the same coefficient γ for the third term yields

$$\begin{aligned}
E_1 &\le \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon}\sum_j\sum_r \delta_{j,r}(p^\varepsilon)^2\\
&\quad + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r l_{jr}|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 - \frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right)\\
&\le \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon}\sum_j\sum_r \delta_{j,r}(p^\varepsilon)^2\\
&\quad + \frac{h}{2\gamma\varepsilon}\sum_j\sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 - \frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right).
\end{aligned}$$

We now look at the last term of this inequality $W = -\frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon, \delta_{j,r}(\mathbf{u}^\varepsilon)\right)$. By definition (60) of $\widehat{\beta}_{j,r}$, one has $\left|\widehat{\beta}_{j,r}\right| \le h^2$. Therefore

$$|W| \le \frac{\sigma h^2}{\varepsilon^2}\left(\sum_j\sum_r\left|\mathbf{u}_r^\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)^{\frac{1}{2}}$$

$$\le \frac{\sigma h^2}{\varepsilon^2}\sqrt{P}\left(\sum_r\left|\mathbf{u}_r^\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)^{\frac{1}{2}} \le \frac{\sigma h}{\varepsilon^2}\sqrt{\frac{P}{C_{\mathcal{M}}}}\left(\sum_r|V_r|\left|\mathbf{u}_r^\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)^{\frac{1}{2}}$$

$$\le \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_{\mathcal{M}}}}\left(\sum_r|V_r|(\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right).$$

It yields

$$\begin{aligned}
E_1 &\le \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon}\sum_j\sum_r \delta_{j,r}(p^\varepsilon)^2\\
&\quad + \left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_{\mathcal{M}}}}\right)\sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_{\mathcal{M}}}}\sum_r|V_r|(\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon).
\end{aligned} \tag{130}$$

Using the first interpolation result of proposition 3.3 and the assumption (65), one has

$$\sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \le PC_\mathcal{A}^2 ||p^\varepsilon||^2_{H^2(\Omega)} \text{ and } \sum_j \sum_r |\delta_{j,r}\mathbf{u}^\varepsilon|^2 \le PC_\mathcal{A}^2 ||\mathbf{u}^\varepsilon||^2_{H^2(\Omega)}.$$

So we obtain

$$\int_0^T E_1 dt \le \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 dt + \frac{PC_\mathcal{A}^2 h}{2\gamma\varepsilon}||p^\varepsilon||^2_{L^2([0,T];H^2(\Omega))}$$
$$+ PC_\mathcal{A}^2\left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_\mathcal{M}}}\right)||\mathbf{u}^\varepsilon||^2_{L^2([0,T];H^2(\Omega))} + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_\mathcal{M}}}\|\mathbf{u}_r^\varepsilon\|^2_{L^2([0,T]\times\Omega)}.$$

Using energy estimate (2) for the the second term of the rhs of the above inequality, (3) for the third term and (76) for the last term, one gets finally

$$\int_0^T E_1(t)dt \le \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \tag{131}$$
$$+\left(T\frac{PC_\mathcal{A}^2 h}{2\gamma\varepsilon} + PC_\mathcal{A}^2\left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_\mathcal{M}}}\right)\frac{\varepsilon^2}{\sigma} + \frac{\sigma h}{2\varepsilon^2}\sqrt{\frac{P}{C_\mathcal{M}}}\frac{\varepsilon^2}{\sigma\alpha}\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^2(\Omega)}.$$

After a convenient definition os the constant $C_1(\gamma)$, it ends the proof. □

Proposition A.2. *There exists a constant C_2 such that the third term in the dissipative identity (82) can be bounded as*

$$\int_0^T E_2(t)dt \le C_2\frac{h}{\varepsilon C_\mathcal{M}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}. \tag{132}$$

Proof. We decompose E_2 in (82) in two terms. Making use of the second set of inequalities of the proposition 3.3 and the assumptions (64) and (65), the first one can be bounded as

$$|A| = \left|\frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon))\right| \le \frac{C_\mathcal{A}P}{\varepsilon}h^2\sum_j |p_j^\varepsilon|\|\mathbf{V}^\varepsilon(t)\|_{H^3(\Omega_j)}.$$

Using the inequality $ab \le \frac{1}{2}(a^2+b^2)$, it yields $|A| \le \frac{C_\mathcal{A}P}{2\varepsilon}h^3\sum_j |p_j^\varepsilon|^2 + \frac{C_\mathcal{A}P}{2\varepsilon}h\sum_j \|\mathbf{V}^\varepsilon(t)\|^2_{H^3(\Omega_j)}$. The assumption (67) yields

$$|A| \le \frac{C_\mathcal{A}P}{2\varepsilon}\frac{h}{C_\mathcal{M}}\|\mathbf{V}_h^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{C_\mathcal{A}P}{2\varepsilon}h\|\mathbf{V}^\varepsilon(t)\|^2_{H^3(\Omega)}.$$

The L^2 stability (76) of the scheme $\mathbf{P}_h^\varepsilon$ shows that

$$\|\mathbf{V}_h^\varepsilon(t)\|_{L^2(\Omega)} \le \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)} \le \|\mathbf{V}^\varepsilon(0)\|_{L^2(\Omega)} + \|\mathbf{V}_h^\varepsilon(0) - \mathbf{V}^\varepsilon(0)\|_{L^2(\Omega)} \le (1 + C_\mathcal{A}h)\|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}$$

where the last inequality comes from the initialization stage (71). With the basic energy estimate (2), and since h is bounded, we obtain

$$\int_0^T |A|dt \le T\left(\frac{C_\mathcal{A}P}{2\varepsilon}\frac{h}{C_\mathcal{M}}(1 + C_\mathcal{A}h) + \frac{C_\mathcal{A}P}{2\varepsilon}h\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}.$$

The second contribution in E_2 is $B = \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}|\left(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\right)$. Almost the same calculations show the bound

$$\int_0^T |B|dt \le T\left(\frac{C_\mathcal{A}P}{2\varepsilon}\frac{h}{C_\mathcal{M}}(1 + C_\mathcal{A}h) + \frac{C_\mathcal{A}P}{2\varepsilon}h\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}.$$

Summing the two contributions, it concludes the proof after a convenient definition of C_2. □

Proposition A.3. *Let $\widehat{\gamma} > 0$ be a number which precise value will be determined further. There exists a constant $C_3(\widehat{\gamma})$ which depends on $\widehat{\gamma}$ such that one has the bound for the last term of the dissipative identity (82)*

$$\int_0^T E_3(t)dt \le \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T \sum_r \sum_j l_{jr}\left(\mathbf{n}_{jr}\,,\ \boldsymbol{u}_r^\varepsilon - \boldsymbol{u}_j^\varepsilon\right)^2 dt + C_3(\widehat{\gamma})\frac{h}{\varepsilon C_{\mathcal{M}}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}. \tag{133}$$

Proof. The definition of E_3 in (82) is

$$\begin{aligned} E_3 = &\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\right)\\ &-\frac{\sigma}{\varepsilon^2}\sum_j\int_{\Omega_j}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon)d\mathbf{x} - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_j\mathbf{u}_r^\varepsilon,\mathbf{u}_r^\varepsilon).\end{aligned}$$

Using the Cauchy-Schwarz inequality on the third term $\int(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon)$, one gets

$$\begin{aligned} E_3 \le &\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\right)\\ &-\frac{\sigma}{\varepsilon^2}\sum_j\frac{1}{|\Omega_j|}\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)^2 - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_j\mathbf{u}_r^\varepsilon,\mathbf{u}_r^\varepsilon),\end{aligned}$$

which can be written

$$E_3 \le -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) - \frac{\sigma}{\varepsilon^2}\sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right)$$

that is

$$\begin{aligned} E_3 \le &-\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right),\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\\ &-\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\,,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\\ &-\frac{\sigma}{\varepsilon^2}\sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right).\end{aligned}$$

One has, using the geometric identity $\sum_r\widehat{\beta}_{jr} = |\Omega_j|I_d$ which can be found in [7, 13],

$$\begin{aligned}&\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\,,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\,,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right)\\ &\qquad = \sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\,,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right).\end{aligned}$$

We thus get after simplification

$$\begin{aligned} E_3 \le &-\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right),\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) && := S_1\\ &-\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\,,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right) && := S_2\end{aligned} \tag{134}$$

We add and subtract at each average on the cell the nodal value. We recall the notation $\delta_{j,r}(\mathbf{u}^\varepsilon) = \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}^\varepsilon(\mathbf{x}_r)$. We get for the term under the first sum in (134)

$$\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right),\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)$$

$$= \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) - \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right), \delta_{j,r}(\mathbf{u}^\varepsilon)\right)$$
$$- \left(\widehat{\beta}_{j,r}\delta_{j,r}(\mathbf{u}^\varepsilon), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) + \left(\widehat{\beta}_{j,r}\delta_{j,r}(\mathbf{u}^\varepsilon), \delta_{j,r}(\mathbf{u}^\varepsilon)\right). \qquad (135)$$

The first of these quantities is purely nodal: one has after summation

$$\sum_j \sum_r \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right)$$
$$= \sum_r \left(B_r\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) \geq \alpha \sum_r |V_r||\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)|^2 \qquad (136)$$

with the help of (70). The second and third term in the identity (135) can be bounded by a Young's inequality with a convenient constant $C = \frac{C_{\mathcal{M}}\alpha}{2P}$ so that all terms containing $\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)$ are controlled by (136). So we obtain concerning S_1 defined in (134)

$$S_1 \leq \left(1 + \frac{2P}{C_{\mathcal{M}}\alpha}\right)\frac{h^2\sigma}{\varepsilon^2}\sum_r\sum_j |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2.$$

Using the first interpolation result stressed in proposition 3.3, one has in dimension two $|\delta_{j,r}(\mathbf{u}^\varepsilon)| \leq C_{\mathcal{A}}\|\mathbf{u}^\varepsilon(t)\|_{H^2(\Omega_j)}$. So, taking into account energy estimate (3) we have for the first term

$$\int_0^T S_1 dt \leq C_{\mathcal{A}}^2 P\left(1 + \frac{2P}{C_{\mathcal{M}}\alpha}\right)h^2\|\mathbf{V}^\varepsilon(0)\|^2_{H^2(\Omega)}.$$

We now consider the second term called S_2 in (134)

$$S_2 = -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right).$$

Using $(\vec{a}\otimes\vec{b}\,\vec{c},\vec{d}) = (\vec{b},\vec{c})(\vec{a},\vec{d})$, one has

$$S_2 = -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j l_{jr}\left((\mathbf{x}_r - \mathbf{x}_j), \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)$$

Using the Young's inequality $ab \leq \frac{\widehat{\gamma}\varepsilon}{2}a^2 + \frac{1}{2\widehat{\gamma}\varepsilon}b^2$, we get

$$\int_0^T S_2 dt \leq \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T\sum_r\sum_j l_{jr}\left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt + \int_0^T\frac{\sigma}{2\widehat{\gamma}\varepsilon^3}\sum_r\sum_j l_{jr}\left((\mathbf{x}_r - \mathbf{x}_j), \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)^2 dt$$

Using one more time the energy estimate (3) the second term in the right hand side of the above inequality is bounded by $\frac{Ph}{2C_{\mathcal{M}}\widehat{\gamma}\varepsilon}\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\Omega)}$. Thus

$$\int_0^T E_3(t)dt \leq \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T\sum_r\sum_j l_{jr}\left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt + P\left\{C_{\mathcal{A}}^2\left(1 + \frac{2P}{C_{\mathcal{M}}\alpha}\right)h^2 + \frac{h}{2C_{\mathcal{M}}\widehat{\gamma}\varepsilon}\right\}\|\mathbf{V}^\varepsilon(0)\|^2_{H^2(\Omega)},$$

which is the expected result after convenient redefinition of the constant in front of the last term. □

End of the proof of the naive estimate of proposition (3.5). One gets

$$\mathscr{E}(T) \leq \mathscr{E}(0) - \frac{1}{\varepsilon}\int_0^T\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \int_0^T E_1(t)dt + \int_0^T E_2(t)dt + \int_0^T E_3(t)dt$$

where integrals are estimated in (129), (132) and (133). Using equation (71), one finds

$$\begin{aligned}\mathscr{E}(T) \leq & C_0^2 h^2 \|\mathbf{V}^\varepsilon(0)\|^2_{H^2(\Omega)} \\ & - \frac{1}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \\ & + \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + C_1(\gamma)\frac{h}{\varepsilon\sqrt{C_\mathcal{M}}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)} \\ & + C_2 \frac{h}{\varepsilon C_\mathcal{M}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)} \\ & + \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T \sum_r \sum_j l_{jr}\left(\mathbf{n}_{jr}\ ,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt + C_3(\widehat{\gamma})\frac{h}{\varepsilon C_\mathcal{M}}\|\mathbf{V}^\varepsilon(0)\|^2_{H^2(\Omega)}.\end{aligned}$$

This estimate is fundamental, since it shows the competition between different kind of error terms and the dissipation of the fluxes. Choosing by example $\widehat{\gamma} < \frac{1}{\sigma}$ and $\gamma < \frac{1}{2}$, all terms like $\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2$ vanish. All other terms can put together as $\mathscr{E}(T) \leq \frac{{}_\downarrow C}{2}\frac{h}{\varepsilon}\|p_0\|^2_{H^4(\Omega)}$, where the constant ${}_\downarrow C$ has , as in 1D, has at most a linear growth in time. It ends the proof of the naive estimate. ☐

B Bihari's inequality and application

We recall a nonlinear generalization of the Gronwall-Bellman inequality known as Bihari's inequality

Lemma B.1. *If*

$$y(t) \leq a + \int_0^t b(s)g(y(s))ds, \tag{137}$$

with a non negative constante, $b(t)$ a positive function and g a positive non decreasing function then, noting by $G(x)$ an antiderivative of $1/g(x)$, one has

$$y(t) \leq G^{-1}\left(G(a) + \int_0^t b(s)ds\right). \tag{138}$$

The proof is trivial by setting $Z = a + \int_0^t b(s)g(y(s))ds$ and and verifying that $Z' \leq bg(Z)$, see [5]. In our work $g(x) = \sqrt{x}$. Moreover a, $b(s)^2$ and y are square of $L^2(\Omega)$ norms. More precisely in our convergence's proofs one ends to inequality of the type

$$\|Y\|^2_{L^2(\Omega)}(t) \leq \|Y\|^2_{L^2(\Omega)}(0) + \int_0^t \|A\|^2_{L^2(\Omega)} + \int_0^t \|B\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)}ds. \tag{139}$$

for Y, A and B functions of $L^2(\Omega$. Thus for all $t \leq T$

$$\|Y\|^2_{L^2(\Omega)}(t) \leq \|Y\|^2_{L^2(\Omega)}(0) + \int_0^T \|A\|^2_{L^2(\Omega)} + \int_0^t \|B\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)}ds, \tag{140}$$

Using the Bihari's inequality (138) and the Cauchy-Schwarz inequality one obtain for all $t \leq T$,

$$\|Y\|^2_{L^2(\Omega)}(t) \leq \frac{1}{2}\left(2\sqrt{\|Y\|^2_{L^2(\Omega)}(0) + \|A\|^2_{L^2([0,T]\times\Omega)}} + \sqrt{t}\sqrt{\int_0^t \|B\|^2_{L^2([0,T]\times\Omega)}ds}\right)^2, \tag{141}$$

and majorizing t by T in the right-hand side

$$\|Y\|^2_{L^2(\Omega)}(t) \leq \frac{1}{2}\left(2\sqrt{\|Y\|^2_{L^2(\Omega)}(0) + \|A\|^2_{L^2([0,T]\times\Omega)}} + \sqrt{T}\|B\|_{L^2([0,T]\times\Omega)}\right)^2. \tag{142}$$

Integrating in time that gives

$$\|Y\|^2_{L^2([0,T]\times\Omega)} \leq \frac{1}{2}T\left(2\sqrt{\|Y\|^2_{L^2(\Omega)}(0)+\|A\|^2_{L^2([0,T]\times\Omega)}}+\sqrt{T}\|B\|_{L^2([0,T]\times\Omega)}\right)^2 \tag{143}$$

We can summarize these calculations by the lemma

Lemma B.2. *If Y, A and B are functions of $L^2(\Omega)$ satisfying (139) then*

$$\|Y\|_{L^2([0,T]\times\Omega)} \leq \sqrt{\frac{T}{2}}\left(2\sqrt{\|Y\|^2_{L^2(\Omega)}(0)+\|A\|^2_{L^2([0,T]\times\Omega)}}+\sqrt{T}\|B\|_{L^2([0,T]\times\Omega)}\right) \tag{144}$$

If $\|A\|_{L^2(\Omega)} \leq C$ et $\|B\|^2_{L^2(\Omega)} \leq C$, with C constant then the right-hande side behaves as $T^{\frac{3}{2}}$ for large time. If $\|A\|_{L^2(\Omega)} \leq C$ or $\int_0^T \|A\|_{L^2(\Omega)} \leq C$ and $\int_0^T \|B\|^2_{L^2(\Omega)} \leq C$, then the right-hand side behaves now as T for large time.

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